Exercise sheet 6

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1. *Define a measure space / probability space to describe two unrelated fair coin tosses. What assumptions are you making in giving the description? Define a sigma-algebra suitable for studying the situation where one can only ask if the two coins have the same side up, or different sides up.*

Proof. We consider the probability space $\Omega = \{HH, HT, TH, TT\}$, where HT denotes the occurrence that the first throw results in heads, the second in tails. We choose the sigma-algebra as $\mathcal{F} = \mathcal{P}(\Omega)$, and the probability measure that assigns probability $|A|/4$ to each set $A \in \mathcal{F}$ (so that each element of Ω has probability $1/4$). This description supposes that one can distinguish which of the two throws is the first and which is the second. If this is not the case, then one might consider the sigma-algebra generated by $\{HH\}, \{TT\}, \{HT, TH\}$, i.e. the smallest sigma-algebra containing the latter sets (and the union, intersections and complements thereof):

$$\mathcal{F} = \{\emptyset, \{HH\}, \{TT\}, \{HT, TH\}, \{HH, TT\}, \{HH, HT, TH\}, \{TT, HT, TH\}, \{HH, TT, HT, TH\}\}.$$

The probability measure is then given by

$$\mathbb{P}[\{HH\}] = \mathbb{P}[\{TT\}] = 1/4, \quad \mathbb{P}[\{HT, TH\}] = 1/2,$$
$$\mathbb{P}[\{HH, HT, TH\}] = \mathbb{P}[\{TT, HT, TH\}] = 3/4, \quad \mathbb{P}[\Omega] = 1.$$

When one can only ask whether the two coins show the same face of not, we might consider the sigma-algebra generated by $\{HH, TT\}, \{HT, TH\}$, i.e.

$$\mathcal{F} = \{\emptyset, \{HH, TT\}, \{HT, TH\}, \{HH, TT, HT, TH\}\}$$

and

$$\mathbb{P}[\{HH, TT\}] = 1/2, \quad \mathbb{P}[\{HT, TH\}] = 1/2,$$
$$\mathbb{P}[\Omega] = 1, \quad \mathbb{P}[\emptyset] = 0.$$

□

Exercise 2. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Prove that if A, B are measurable sets, then so is also $A \setminus B := \{a \in A, a \notin B\}$.*

Proof. We first notice that

$$A \setminus B = \{a \in A, a \notin B\} = \{a \in A\} \cap \{a \notin B\} = A \cap B^c.$$

By De Morgan's laws, we might rewrite $A \cap B^c = (A^c \cup B)^c$. Now because A and B are measurable, by definition of a sigma-algebra, A^c is also measurable, together with $A^c \cup B$. It finally follows that $A \cap B^c$ is measurable as the complement of $A^c \cup B$. □

Exercise 3. *Show that the Borel $\sigma-$algebra on $\mathbb{R}^n$ also contains all products of half-lines $\Pi_{i=1}^n(-\infty, a_i]$, all open balls $B(x, r)$ and in fact all open sets of $\mathbb{R}^n$.*

Proof. Note that we can write

$$\prod_{i=1}^n (-\infty, a_i] = \bigcup_{k\geq 1} \prod_{i=1}^n [-k, a_i],$$

which is a countable union of measurable sets and hence measurable. Furthermore we have that all open boxes are of the form

$$\prod_{i=1}^n (a_i, b_i) = \bigcup_{k\geq 1} \prod_{i=1}^n (-\infty, a_i]^c \cap \prod_{i=1}^n (-\infty, b_i - 1/k]$$

and hence measurable. For a general open set U, we claim that we can write:

$$U = \bigcup_{(a_i)_{i=1}^n, (b_i)_{i=1}^n \in \mathbb{Q}^n : \prod_{i=1}^n (a_i, b_i) \subset U} \prod_{i=1}^n (a_i, b_i).$$

Indeed, the union is non-empty because U is open: for every $x \in U$, there exists δ small enough so that $\prod_{i=1}^n (x_i - \delta, x_i + \delta) \subset U$ (and up to taking a smaller δ we may assume that it is rational). The inclusion from right to left clearly holds, and for the other one, notice that for each $x \in U$ with δ as above such that $\prod_{i=1}^n (x_i - \delta, x_i + \delta) \subset U$ we can take a point y in $\prod_{i=1}^n (x_i - \delta, x_i + \delta)$ with rational coordinates at distance $\leq \delta/3$ from x, and then $\prod_{i=1}^n [y_i - \delta/2, y_i + \delta/2]$ contains x and is included in U. We conclude that U is measurable as a countable union of measurable sets. □

Exercise 4. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Prove that if $A_1 \subseteq A_2 \subseteq A_3 \dots$ are an increasing sequence of measurable sets, then $\mu(\bigcup_{i\geq 1} A_i) = \lim_{i\to\infty} \mu(A_i)$.*

Prove also that if $A_1, A_2, \dots$ are any measurable sets, then the so called union bound $\mu(\bigcup_{i\geq 1} A_i) \leq \sum_{i\geq 1} \mu(A_i)$ holds. Interpret it in the probabilistic context.

Proof. Consider for all $n \geq 1$, $B_n := A_n \setminus A_{n-1}$, and $B_1 = A_1$. Now $(B_n)_{n\geq 1}$ is a collection of disjoint measurable sets, as for $1 \leq i < j$, $B_i \subset A_{j-1}$ and $A_{j-1} \cap B_j = \emptyset$. Note also that $\bigcup_{n\geq 1} A_n = \bigcup_{n\geq 1} B_n$. Therefore by countable additivity of μ, it holds that[1] (defining $A_0 := \emptyset$ for a more compact notation)

$$\begin{aligned}
\mu(\bigcup_{n\geq 1} A_n) = \mu(\bigcup_{n\geq 1} B_n) &= \sum_{n\geq 1} \mu(B_n) \\
&= \sum_{n\geq 1} \mu(A_n \setminus A_{n-1}) \\
&= \lim_{N\to\infty} \sum_{n=1}^N \mu(A_n \setminus A_{n-1}) \\
&= \lim_{N\to\infty} \sum_{n=1}^N [\mu(A_n) - \mu(A_{n-1})] \\
&= \lim_{N\to\infty} \mu(A_N)
\end{aligned}$$

where we have used that

$$\mu(A_n) = \mu((A_n \setminus A_{n-1}) \cup A_{n-1}) = \mu(A_n \setminus A_{n-1}) + \mu(A_{n-1})$$

by the finite additivity property of μ.

[1]Note as well that the limit at the end is well-defined as $\mu(A_n)$ is an increasing sequence, given that the sets $(A_n)_{n\geq 1}$ are nested/increasing

For the union bound, define for $n \geq 1$ the sets

$$B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i,$$

such that $B_i \cap B_j = \emptyset$ for $i \neq j$. It holds that $\cup_{n\geq 1} B_n = \cup_{n \geq 1} A_n$, given that the left inclusion is trivial, and that for $x \in \cup_{n\geq 1} A_n$, we have that $x \in B_k$ for $k = \min\{i \geq 1 : x \in A_i\} < +\infty$. By countable additivity of μ, we then obtain:

$$\begin{aligned}
\mu\left(\bigcup_{n\geq 1} A_n\right) = \mu\left(\bigcup_{n\geq 1} B_n\right) &= \sum_{n=1}^{+\infty} \mu(B_n) \\
&= \sum_{n=1}^{+\infty} \mu\left(A_n \setminus \bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{n=1}^{+\infty} \mu(A_n)
\end{aligned}$$

by monotonicity of μ. This bound can be interpreted in the following way: the probability that a "bad event" A_n happens for some $n \in \mathbb{N}$ is less than the sum of the individual probability that each A_n happens, with equality only when the events are mutually exclusive. □

Exercise 5. *Show that the Lebesgue measure of* $\mathbb{R}^n$ *is infinite and that the Lebesgue measure of a line segment* $[0,1] \subseteq \mathbb{R}^n$ *is zero.*

Now consider the Lebesgue measure on $\mathbb{R}$. *Prove that the measure of irrational numbers contained in* $[0,R]$ *is equal to R; prove also that the Lebesgue measure of the Cantor set is zero.*

Proof. For any $k \in \mathbb{N}$ we have $\prod_{i=1}^{n}[0,k] \subset \mathbb{R}^n$ and hence $\lambda(\mathbb{R}^n) \geq \lambda(\prod_{i=1}^{n}[0,k]) = k^n \to +\infty$, for $k \to +\infty$. For the line segment and any $\epsilon > 0$ we have $\prod_{i=1}^{n-1}\{0\} \times [0,1] \subset \prod_{i=1}^{n-1}[\epsilon/2, \epsilon/2] \times [0,1]$ and hence

$$\lambda(\prod_{i=1}^{n-1}\{0\} \times [0,1]) \leq \lambda(\prod_{i=1}^{n-1}[\epsilon/2, \epsilon/2] \times [0,1]) = \epsilon^{n-1},$$

which can be made arbitrarily small. For the measure of irrational numbers (which are measurable as the complement of the rational numbers), note that $[0,R] = ([0,R] \cap \mathbb{Q}) \cup ([0,R] \cap (\mathbb{R} \setminus \mathbb{Q}))$ which implies

$$R = \lambda([0,R]) = \lambda([0,R] \cap \mathbb{Q}) + \lambda([0,R] \cap (\mathbb{R} \setminus \mathbb{Q})) = \lambda([0,R] \cap (\mathbb{R} \setminus \mathbb{Q})),$$

where in the last equation we used that the rational numbers have measure zero. Finally, recall that we can write the cantor set as $C = \bigcap_{n\geq 1} C_n$, where each C_n contains 2^n closed and disjoint intervals of length $(1/3)^n$. This implies that, for any $n \in \mathbb{N}$

$$\lambda(C) = \lambda(\cap_{m\geq 1} C_m) \leq \lambda(C_n) = \left(\frac{2}{3}\right)^n,$$

which can be made arbitrarily small.

□

Exercise 6. *Show that there is no finite, non-zero measure on* $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ *that is translation-invariant, i.e. such that* $\mu(A+n) = \mu(A)$ *for all* $n \in \mathbb{N}$ *and* $A \in \mathcal{P}(\mathbb{N})$.

Proof. Assume for the sake of contradiction, that such a measure μ exists. By assumption, the singletons $\{n\}$ are measurable for all $n \in \mathbb{N}$. Choosing $x = 1$ in our translation-invariance assumption, we see that every singleton has the same measure

$$\mu\{1\} = \mu\{n\}.$$

Using σ-additivity, we have that

$$\mu(\mathbb{N}) = \sum_{n\geq 1} \mu\{n\} = \infty \cdot \mu\{1\} = \begin{cases} 0 & \mu\{1\} = 0 \\ \infty & \mu\{1\} > 0. \end{cases}$$

But we assumed our measure μ to be finite and non-zero – contradicting the above calculations. $\square$