Exercise sheet 5

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1. *Find a sequence* $(f_n)_{n\geq 1} \subset C^1([0,1],\mathbb{R})$ *that converges to some* $f \in C([0,1],\mathbb{R})$ *with respect to the uniform norm* $\|\cdot\|_\infty$*, but where* f *is not differentiable. Now show that if convergence holds w.r.t. the norm* $\|f\|_\infty + \|f'\|_\infty$*, then the limit is also continuously differentiable.*

Proof. Let $g \mapsto |||g||| := \|g\|_\infty + \|g'\|_\infty$, whenever the expression is well-defined and finite (the suprema are taken over the closed unit interval K). Note first that $|||\cdot|||$ is indeed a norm: it is indeed finite and positive with $|||g||| = 0$ iff $g \equiv 0$ (and $g' \equiv 0$), and the triangular inequality and scaling are immediate from $\|\cdot\|_\infty$ (and linearity of the differentiation).

Observe that if a function belongs to the normed space $(\mathcal{C}(K), |||\cdot|||)$, it automatically belongs to the normed space $(\mathcal{C}(K), ||\cdot||_\infty)$.

The sequence of functions

$$f_n(x) := \sqrt{(x-1/2)^2 + 1/n}$$

clearly belong to $(\mathcal{C}(K), |||\cdot|||)$ and converges in $(\mathcal{C}(K), ||\cdot||_\infty)$ to $f := |x - 1/2|$. However, we see that convergence in $(\mathcal{C}(K), |||\cdot|||)$ is not possible, since the sequence of derivatives

$$f_n'(x) := \frac{x - 1/2}{\sqrt{(x-1/2)^2 + 1/n}}$$

does not converge uniformly, and therefore (f_n) cannot converge in $(\mathcal{C}(K), |||\cdot|||)$. Indeed, the pointwise limit is equal to a translated sign function g defined on K as

$$g(x) := \begin{cases} -1 & x < 1/2 \\ 0 & x = 0 \\ +1 & x > 1/2, \end{cases}$$

which is discontinuous, so the convergence of f_n' to g cannot be uniform, as $(\mathcal{C}(K), ||\cdot||_\infty)$ is closed (as a Banach space).

Now, consider $(f_n)_{n\geq 1} \subset C^1(D,\mathbb{R})$ converging to $f \in C(D,\mathbb{R})$ with respect to the norm $|||\cdot|||$. If $|||f_n - f||| \to 0$ as $n \to \infty$, in particular $\|f_n' - f'\|_\infty \to 0$, so f is differentiable and $(f_n')_{n\geq 0}$ converges uniformly to f': it results that f' is continuous. □

Exercise 2. *Show that the Riemann integral satisfies some desirable properties:*

- *All continuous functions on* $[0,1]$ *are Riemann integrable*
- *Every piecewise constant function is Riemann integrable*
- *Linearity: If* f, g *are Riemann integrable on* $[0,1]$*, then so is their sum and the integral is equal to the sums.*

Proof. Throughout this exercise we rely on Theorem 9 from the Reminder sheet.

1. Let $f : [0,1] \to \mathbb{R}$ be continuous. As $[0,1]$ is compact, f is uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. In particular, considering the partition $P_n := \{i2^{-n}\}_{i=0}^{2^n}$ for n large enough so that $2^{-n} < \delta$, for all $0 \leq i \leq 2^n - 1$,

$$\sup_{x\in[i2^{-n},(i+1)2^{-n}]} f(x) - \inf_{x\in[i2^{-n},(i+1)2^{-n}]} f(x) < \varepsilon.$$

We now compute:

$$U(P_n,f)-L(P_n,f) = 2^{-n}\sum_{i=0}^{2^n-1}\left(\sup_{x\in[i2^{-n},(i+1)2^{-n}]} f(x) - \inf_{x\in[i2^{-n},(i+1)2^{-n}]} f(x)\right) \leq 2^{-n}\sum_{i=0}^{2^n-1}\varepsilon = \varepsilon.$$

The claim follows by arbitrariness of $\varepsilon > 0$.

2. Let $f : [0,1] \to \mathbb{R}$ be piecewise constant, i.e. it changes value finitely many times with $0 = x_0 < x_1 < \cdots < x_m = 1$ being the points of discontinuity, and f is constant equal to c_j on intervals (x_j, x_{j+1}) (with $f(x_j) = c_j$ or c_{j-1}). Thus, within these intervals, we have $\sup f = \inf f$. Let $\varepsilon > 0$ and consider the partition

$$P_n = \left\{0, 1, x_j \pm 2^{-n}, 1 \leq j \leq m-1\right\},$$

ordered by increasing value, and with n large enough so that $2^{-n+1} < \min_{0\leq j\leq m-1}|x_{j+1}-x_j|$ (draw a sketch!). As observed above, the only intervals of the partition over which the supremum and the infimum of the function differ are those of the form $[x_j - 2^{-n}, x_j + 2^{-n}]$. We get that

$$U(P_n,f) - L(P_n,f) \leq \sum_{j=0}^{m-2} 2^{-n+1}|c_{j+1} - c_j| \leq \varepsilon$$

provided that n is large enough, showing f is integrable.

3. Let f, g be integrable on $[0,1]$: by Theorem 9 again, for $\varepsilon > 0$ there are partitions P_f, P_g such that $U(P_f, f) - L(P_f, f) < \varepsilon$, $U(P_g, g) - L(P_g, g) < \varepsilon$. Consider the partition $P = P_f \cup P_g$, reordered by increasing value. Since P is a refinement of P_f and P_g, it holds that

$$U(P,f) \leq U(P_f,f), L(P,f) \geq L(P_f,f), \quad U(P,g) \leq U(P_g,g), L(P,g) \geq L(P_g,g).$$

Furthermore, writing P as $P = \{0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1\}$, then

$$\begin{aligned}
U(P, f+g) &= \sum_{i=0}^{n-1}(x_{i+1}-x_i)\sup_{[x_i,x_{i+1}]}|f+g| \\
&\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left(\sup_{[x_i,x_{i+1}]}|f| + \sup_{[x_i,x_{i+1}]}|g|\right) \\
&\leq U(P,f) + U(P,g)
\end{aligned}$$

and similarly $L(P, f+g) \geq L(P,f) + L(P,g)$. Gathering everything, we find that

$$\begin{aligned}
U(P,f+g) - L(P,f+g) &\leq U(P,f) + U(P,g) - L(P,f) - L(P,g) \\
&\leq (U(P_f,f) - L(P_f,f)) + (U(P_g,g) - L(P_g,g)) < \varepsilon
\end{aligned}$$

showing that $f+g$ is Riemann integrable.

□

Exercise 3. *We aim to conclude the proof of Proposition 1.9 in the notes. Recall that the setup consisted of $f \in C^2([0,1],\mathbb{R})$ twice continuously differentiable and satisfying $f(0) = f(1)$ and $f'(0) = f'(1)$, and that we argued in the first part of the proof that*

$$f_N := \sum_{n=1}^{N}(s_n \sin(2\pi nx) + c_n\cos(2\pi nx)),$$

converges w.r.t. $\|\cdot\|_\infty$ *as* $N \to \infty$ *to some function* $g \in C([0,1],\mathbb{R})$*, where* $(s_n)_{n\geq 1}$ *and* $(c_n)_{n\geq 1}$ *are the Fourier coefficients of* f*.*

Using the definition of f_N *and* g *show that for all* $n \geq 0$*,*

$$\int_0^1 (f-g)\cos(2\pi nx)\mathrm{d}x = \int_0^1 (f-g)\sin(2\pi nx)\mathrm{d}x = 0,$$

and conclude the proof with the help of Proposition 1.12.

Proof. The critical thing to recall from the lecture is that the limit g is a uniform limit of the continuous functions f_N, and can therefore be swapped with the integral (cf. Theorem 10 in the reminders sheet[1]), together with the integrals computed in sheet 3 exercise 3. Write $K := [0,1]$. We have

$$\begin{aligned}\int_K g(x)\cos(2\pi mx)\mathrm{d}x &= \lim_{N\to\infty}\sum_{n=1}^{N}\int_K (s_n\sin(2\pi nx) + c_n\cos(2\pi nx))\cos(2\pi mx)\mathrm{d}x \\ &= \lim_{N\to\infty}\sum_{n=1}^{N}(0 + \frac{1}{2}c_n\delta_{n,m}) \\ &= \frac{1}{2}c_m,\end{aligned}$$

yielding the first integral is equal to zero. Analogous computations show that the second integral vanishes. $\square$

Exercise 4 (Fejér kernel, I)**.** *One possible choice for the function* T_{n,x_0} *in the notes is the so called Fejér kernel, denoted* $F_n^{x_0}$*. The Fejér kernel for* $x_0 = 0$ *is given by*

$$F_n^0(x) = 1 + \sum_{k=1}^{n-1} 2\left(1 - \frac{k}{n}\right)\cos(2\pi kx).$$

Deduce the expression for $F_n^{x_0}$ *for* $x_0 \in (0,1)$*.*

Now prove carefully the following properties, thereby proving Lemma 1.13:

1. $\forall n \geq 1,\ F_n^{x_0}(x) \geq 0,$
2. $\forall n \geq 1,\ \int_0^1 F_n^{x_0}(x)\mathrm{d}x = 1,$
3. $\forall \varepsilon > 0, \lim_{n\to\infty}\int_0^1 \mathbf{1}_{|x-x_0|>\varepsilon}F_n^{x_0}(x)\mathrm{d}x = 0.$

Proof. Since for $x_0 = 0$ we have

$$F_n^0(x) = 1 + \sum_{k=1}^{n-1} 2\left(1 - \frac{k}{n}\right)\cos(2\pi kx),$$

and for arbitrary centers $x_0 \in (0,1)$, we just shift the kernel by replacing x with $x - x_0$. Thus, we define

$$F_n^{x_0}(x) = F_n^0(x - x_0) = 1 + \sum_{k=1}^{n-1} 2\left(1 - \frac{k}{n}\right)\cos\bigl(2\pi k(x - x_0)\bigr).$$

[1] For the sake of completeness, we include a short proof: let f be the uniform limit of a sequence of continuous functions (f_n) on a bounded set K. In particular it is continuous, hence Riemann integrable. Then,

$$\int_K f dx = \lim_{n\to\infty}\int_K f_n dx.$$

Indeed, using the triangle inequality of integrals, we have

$$\left|\int_K f dx - \int_K f_n dx\right| = \left|\int_K (f - f_n)dx\right| \leq \mathrm{diam}(K)||f - f_n||_\infty \to 0.$$

1. We write

$$\begin{aligned}
F_n^0(x) &= 1 + \sum_{k=1}^{n-1} 2\left(1 - \frac{k}{n}\right)\cos(2\pi k x) \\
&= \frac{1}{n}\sum_{k=-n+1}^{n-1}(n - |k|)e^{2\pi i k x} \\
&= \frac{1}{n}(e^{2\pi i x \frac{n-1}{2}} + e^{2\pi i x \frac{n-3}{2}} + \cdots + e^{-2\pi i x \frac{n-1}{2}})^2 \\
&= \frac{1}{n}\left(e^{-\pi i (n-1) x}\frac{e^{2\pi i n x} - 1}{e^{2\pi i x} - 1}\right)^2 \\
&= \frac{1}{n}\left(\frac{\sin(\pi n x)}{\sin(\pi x)}\right)^2.
\end{aligned}$$

In general, we obtain

$$F_n^{x_0}(x) = \frac{1}{n}\left(\frac{\sin(\pi n(x - x_0))}{\sin(\pi(x - x_0))}\right)^2 \geq 0. \tag{1}$$

2. Now, using the definition of $F_n^{x_0}$, the linearity of the integral, and

$$\int_0^1 \cos(2\pi k(x - x_0))\,\mathrm{d}x = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases}$$

we get

$$\int_0^1 F_n^{x_0}(x)\,\mathrm{d}x = \int_0^1 1\,\mathrm{d}x + \sum_{k=1}^{n-1} 2\left(1 - \frac{k}{n}\right)\int_0^1 \cos(2\pi k x)\mathrm{d}x = 1.$$

3. Since $x_0 \in (0,1)$, for $\varepsilon < \min(x_0, 1 - x_0)$ and $|x - x_0| > \varepsilon$ we have that

$$|\sin(\pi(x - x_0))| \geq \frac{\pi}{2}\varepsilon$$

(this follows for instance from $|\sin(x)| \geq |x|/2$ on $[-\pi/2, \pi/2]$). Applying it to Equation (1), we get that

$$F_n^{x_0}(x) \leq \frac{4}{n\pi^2\varepsilon^2}$$

for x such that $|x - x_0| > \varepsilon$, so that

$$\int_0^1 \mathbf{1}_{|x - x_0| > \varepsilon} F_n^{x_0}(x)\mathrm{d}x \leq \frac{4}{n\pi^2\varepsilon^2}.$$

□

Exercise 5 (Fejér kernel, II)**.**

1. *For $f \in C([0,1], \mathbb{R})$ such that $f(0) = f(1)$, show that*

$$\int_0^1 f(x)F_n^{x_0}(x)\mathrm{d}x \xrightarrow[n\to\infty]{} f(x_0).$$

Therefore, the Fejér kernel is in some sense an 'approximate Dirac delta function'.

2. *Compute formally the Fourier coefficients of the Dirac delta function, i.e. a 'function'*[2] $\delta_{x_0} : [0,1] \to \mathbb{R}^+$ *for* $x_0 \in [0,1]$ *fixed such that for all* $f \in C([0,1],\mathbb{R})$ *such that* $f(0) = f(1)$,

$$\int_0^1 f(x)\delta_{x_0}(x)\mathrm{d}x = f(x_0),$$

and write formally its Fourier series.

3. *Show that the Fourier coefficients of* $F_n^{x_0}$ *converge to the formal expression that you found in 2.*

Proof. 1. Since f is continuous on $[0,1]$ and $f(0) = f(1)$, we may view f as a continuous 1-periodic function, and uniformly continuous: for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| \le \varepsilon$ whenever $|x - y| \le \delta$. Using that the Fejér kernel integrates to one (see the previous exercise) we may rewrite:

$$\begin{aligned}\int_0^1 f(x)F_n^{x_0}(x)\mathrm{d}x &= \int_0^1 (f(x) - f(x_0))F_n^{x_0}(x)\mathrm{d}x + f(x_0)\\ &= \int_0^1 \mathbf{1}_{|x-x_0|>\delta}(f(x) - f(x_0))F_n^{x_0}(x)\mathrm{d}x + \int_0^1 \mathbf{1}_{|x-x_0|\le\delta}(f(x) - f(x_0))F_n^{x_0}(x)\mathrm{d}x + f(x_0)\end{aligned}$$

We estimate the two integrals on the right-hand side. First,

$$\left|\int_0^1 \mathbf{1}_{|x-x_0|>\delta}(f(x) - f(x_0))F_n^{x_0}(x)\mathrm{d}x\right| \le 2\|f\|_\infty \int_0^1 \mathbf{1}_{|x-x_0|>\delta}F_n^{x_0}(x)\mathrm{d}x \xrightarrow[n\to\infty]{} 0$$

by the third property shown in the previous exercise of $F_n^{x_0}$ and the fact that f is bounded. On the other hand, by continuity,

$$\int_0^1 \mathbf{1}_{|x-x_0|\le\delta}\big|f(x) - f(x_0)\big|\, F_n^{x_0}(x)\mathrm{d}x \le \varepsilon \int_0^1 F_n^{x_0}(x)\mathrm{d}x = \varepsilon.$$

This concludes the proof as it shows that

$$\limsup_{n\to\infty}\left|\int_0^1 f(x)F_n^{x_0}(x)\mathrm{d}x - f(x_0)\right| \le \varepsilon$$

and ε was arbitrary.

2. To compute the Fourier coefficients of δ_{x_0}, we formally compute:

$$\int_0^1 \cos(2\pi nx)\,\delta_{x_0}(x)dx = \cos(2\pi nx_0), \quad \int_0^1 \sin(2\pi nx)\delta_{x_0}(x)dx = \sin(2\pi nx_0), \quad \int_0^1 \delta_{x_0}(x)dx = 1,$$

so that formally

$$\delta_{x_0}(x) = 1 + \sum_{n=1}^{\infty}\Big[2\,\cos(2\pi nx_0)\,\cos(2\pi nx) + 2\,\sin(2\pi nx_0)\,\sin(2\pi nx)\Big].$$

On the other hand, the Fejér kernel can be rewritten (using a trigonometric identity for $\cos(2\pi k(x - x_0))$)

$$F_n^{x_0}(x) = 1 + \sum_{k=0}^{n-1}\left(2\left(1 - \frac{k}{n}\right)\cos(2\pi kx_0)\cos(2\pi kx) + 2\left(1 - \frac{k}{n}\right)\sin(2\pi kx_0)\sin(2\pi kx)\right).$$

It follows that for each $k \ge 0$, the coefficients $c_k(F_n^{x_0}), s_k(F_n^{x_0})$ tend to the respective Fourier coefficients $c_k(\delta_{x_0}), s_k(\delta_{x_0})$ as $n \to \infty$.

□

[2] Such a function does not exist, in the classical sense!

Exercise 6 (The Cantor set)**.** *Consider the following iteration: we set $C_0 = [0,1]$ and obtain C_1 by removing the middle third, i.e. $C_1 = C_0 \setminus (1/3, 2/3)$. Now to obtain C_2, we remove the middle third of the both remaining intervals. We continue iteratively and define $C = \cap_{i\geq 1} C_i$. Prove that C is a closed set (i.e. $[0,1]\setminus C$ is open) with empty interior (i.e. there is no open interval contained in C).*

It also has uncountably many elements and is a perfect set - both of these are in the for fun section. It can also be described as the set of numbers in $[0,1]$ that admit a representation in ternary expansion containing no occurrence of 1.

Proof. Let us start by noticing that one can write:

$$C_n = C_{n-1} \cap \bigcup_{k=0}^{3^{n-1}-1} \left[\frac{3k}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right],$$

so that each C_n is closed by induction. It follows that $C = \bigcap_{n\geq 1} C_n$ is closed as an intersection of closed sets.

For the second part, suppose there existed $U \subset C$ open and non-empty, and let $x \in U$: by assumption there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subset U$. Taking n large enough such that $2/3^n < \delta$, we would then have that $(x-\delta, x+\delta) \subset U \subset C \subset C_n$, but C_n is a disjoint union of intervals of length $2/3^n < \delta$, also at distance $1/3^n$ away from eachother, contradiction: we deduce that $\mathrm{int}(C) = \emptyset$. □