Exercise sheet 4

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Spaces of continuous functions II

Exercise 1. *When D is closed and bounded, is the uniform norm the only norm we can put on $(C(D,\mathbb{R}),+)$ such that completeness holds?*

Proof. It is not: consider for example $|||f||| := \sup_{x\in D} |(1+|x|)f(x)| = \|(1+|\cdot|)f\|_\infty$. Since D is bounded, there exists $c>0$ such that every $x \in D$ is such that $|x| \le c$, and we obtain that for all $f \in C(D,\mathbb{R})$,

$$\|f\|_\infty \le |||f||| \le (c+1)\|f\|_\infty,$$

so that $\|\cdot\|_\infty$ and $|||f|||$ are equivalent. We claim that the completeness of $(C(D,\mathbb{R}),\|\cdot\|_\infty)$ implies the completeness of $(C(D,\mathbb{R}),|||\cdot|||)$. Indeed, let $(f_n)_{n\ge1} \subset C(D,\mathbb{R})$ be a Cauchy sequence for $|||\cdot|||$: for all $\varepsilon>0$, there exists $N\ge1$ such that for all $m,n\ge N$, $|||f_n - f_m||| \le \varepsilon$, meaning that $\|f_n - f_m\|_\infty \le \varepsilon$: therefore $(f_n)_{n\ge1}$ is Cauchy in $(C(D,\mathbb{R}),\|\cdot\|_\infty)$, which is complete, and there exists an element $f \in C(D,\mathbb{R})$ with $\|f_n - f\|_\infty \xrightarrow[n\to\infty]{} 0$. It follows that $|||f_n - f||| \xrightarrow[n\to\infty]{} 0$ too and this shows that $(C(D,\mathbb{R}),|||\cdot|||)$ is complete. □

Remark: The norm $|||\cdot|||$ can be viewed as re-weighting f.

Exercise 2. *Find an example of D that is not closed or not bounded, and $f\in C(D,\mathbb{R})$ such that $\|f\|_\infty$ is infinite.*

Show that if we define a distance $\hat{d}_\infty(f,g) := \min(\|f-g\|_\infty, 1)$ on $C(D,\mathbb{R})$ where $D\subset\mathbb{R}$ is arbitrary, then $(C(D,\mathbb{R}),\hat{d}_\infty)$ is still complete. Argue that $\hat{d}_\infty$ can not be represented by a norm, i.e. there is no norm $\|\cdot\|$ such that $\|f\| = \hat{d}_\infty(0,f)$.

Proof. For the case where D is not bounded, consider $D=\mathbb{R}$ and $f : x \mapsto x$, then $f\in C(\mathbb{R},\mathbb{R})$ but clearly $\|f\|_\infty$ is infinite. For the case where D is not closed, consider $D=(0,1)$ and $f : x\mapsto \frac{1}{x}$. We have $f\in C((0,1),\mathbb{R})$ and again $\|f\|_\infty$ is infinite. For the second part of the exercise, let $(f_n)_{n\ge1}$ be a Cauchy sequence in $(C(D,\mathbb{R}),\hat{d}_\infty)$, i.e., for every $\epsilon>0$ there is an n_ϵ such that for all $n,m\ge n_\epsilon$ we have $\hat{d}_\infty(f_n,f_m) = \min(\|f_n-f_m\|_\infty,1) < \epsilon$. This clearly implies that $(f_n)_{n\ge1}$ is also a Cauchy sequence w.r.t. $\|\cdot\|_\infty$. One can then replicate fac simile the 3ε-argument from the class that showed that $C(D,\mathbb{R})$ is complete when D is compact[1]: we find $f\in C(D,\mathbb{R})$ such that $\lim_{n\to\infty}\|f_n - f\|_\infty = 0$, which implies $\lim_{n\to\infty}\hat{d}_\infty(f_n,f)=0$ and shows that $(C(D,\mathbb{R}),\hat{d}_\infty)$ is complete.

The fact that $\hat{d}_\infty$ cannot be represented by a norm can be seen by the following example. Consider the constant function $f=1$, which takes value 1 everywhere. Then

$$\|2f\| = \hat{d}_\infty(0,2f) = \min(\|2f\|_\infty,1) = \min(2\cdot\|f\|_\infty,1) = \min(2,1) = 1 \ne 2 = 2\cdot\|f\|.$$

hence the homogeneity condition is not satisfied. □

Exercise 3 (Peano curve)**.**

1. *Show that the sequence $(f_n)_{n\ge1}$ defined as above is Cauchy, and therefore converges uniformly to a continuous function $f:[0,1]\mapsto[0,1]^2$.*

[1] This proof did not use the fact that D is compact, other than to obtain finiteness of the norm.

2. *Show that f is surjective.*

Sketch. 1) We show that the difference between two sequence elements becomes arbitrarily small. Without loss of generality, let $n \geq m$ and observe that each individual step refines the previous step with movements of at most 3^{-n} in the supremum norm. This means that if we instead compare the n-th and m-th step as above, we get

$$\|f_n - f_m\|_\infty \leq 3^{-\min(m,n)}.$$

The sequence (f_n) belongs to the Banach space $\mathcal{C}([0,1];\mathbb{R}^2)$ and therefore, by completeness, it converges to some $f \in \mathcal{C}([0,1];\mathbb{R}^2)$.

2) Let $x \in [0,1]^2$ be any point in the unit square. Recall that we define the distance between a point x and a set A as

$$d(x, A) := \inf_{a \in A} d(x, a),$$

that is, the smallest distance between x and the set A. Notice in our case A is the image of a function, so that

$$d(x, \mathrm{Im} f) = \inf_{y \in [0,1]} d(x, f(y))$$

Observe that by construction, for each $n \geq 1$, there exists $y_n \in [0,1]$ such that $d(x, f_n(y_n)) \leq 3^{-n}$. Since $(y_n)_{n\geq 1} \subset [0,1]$, by Bolzano-Weierstrass there is a subsequence $(y_{n_k})_{k\geq 1}$ and $y \in [0,1]^2$ such that $y_{n_k} \xrightarrow[k\to\infty]{} y$, and $f(y_{n_k}) \xrightarrow[k\to\infty]{} f(y)$ by continuity. Gathering everything, we obtain that

$$\begin{aligned} d(x, f(y)) &\leq d(x, f_{n_k}(y_{n_k})) + d(f_{n_k}(y_{n_k}), f(y_{n_k})) + d(f(y_{n_k}), f(y)) \\ &\leq 3^{-n_k} + \|f_{n_k} - f\|_\infty + d(f(y_{n_k}), f(y)) \end{aligned}$$

Hence, taking $k \to \infty$, we obtain that $d(x, f(y)) = 0$, showing that $x \in \mathrm{Im}\, f$.

□

Fourier II

Exercise 4. *Consider the function $f : x \mapsto x$ on $[0,1]$.*

1. *Show that the Fourier coefficients of f are $c_0 = 1/2$, $c_n = 0$, $s_n = -1/(\pi n)$ for $n \geq 1$.*
2. *Using that*

$$\sum_{n=1}^{+\infty} \frac{z^n}{n} = -\log(1-z)$$

for $z \in \mathbb{D} \setminus \{1\}$ (the unit disk in $\mathbb{C}$ minus 1), show that $\sum_{n\geq 1}^{+\infty} \sin(2\pi n x)/(\pi n)$ converges and satisfies the identity

$$f(x) = x = \frac{1}{2} - \sum_{n=1}^{+\infty} \frac{\sin(2\pi n x)}{\pi n}.$$

3. *Is the identity still valid at $x = 0$ or $x = 1$?*

Proof. 1) We compute

$$c_0 = \int_0^1 x\, dx = \frac{1}{2}.$$

and for $n \geq 1$

$$c_n = 2\int_0^1 x \cos(2\pi n x)\, dx = 0,$$

using integration by parts. For the sine coefficients s_n:

$$s_n = 2\int_0^1 x\sin(2\pi n x)\,dx = -\frac{1}{\pi n}.$$

2) We have that[2]

$$\sum_{n\geq 1}^{N} \frac{\sin(2\pi n x)}{\pi n} = \frac{1}{\pi}\,\mathrm{Im}\Big(\sum_{n\geq 1}^{N} \frac{e^{2\pi i n x}}{n}\Big),$$

and since the right-hand sum converges as $N \to \infty$ for $x \in (0,1)$ by the hint, the left-hand side must converge as well and the series can be expressed as

$$\sum_{n\geq 1}^{+\infty} \frac{\sin(2\pi n x)}{\pi n} = \frac{1}{\pi}\,\mathrm{Im}(-\log\big(1-e^{2\pi i x}\big)) = -\frac{1}{\pi}\arg(1-e^{2\pi i x}) = \frac{1}{2} - x,$$

using that

$$1 - e^{2\pi i x} = e^{\pi i x}(e^{-\pi i x} - e^{\pi i x}) = -2i\sin(\pi x)e^{\pi i x} = \sin(\pi x)e^{i(\pi x - \pi/2)}.$$

3) At $x = 0$ and $x = 1$, the series clearly converges to $1/2$, different from the value of f at these points.

The points $x = 0, 1$ are discontinuities for the 1-periodic extension of f over the real line. That we cannot have pointwise continuity of the Fourier series at a jump can be intuitively seen from the fact that the approximations made by trigonometric polynomials are continuous and therefore cannot converge uniformly to a non-continuous function. It is not a coincidence that $\mathcal{F}f$ at these two discontinuity points equals to the average of the values of f from the right and to the left of $x = 0$ and $x = 1$, respectively. A theorem due to Lebesgue states that this is always the case whenever we have a discontinuity point.

□

[2] Note: $\mathrm{Im}(\cdot)$ denotes here the imaginary part of a complex number.