Exercise sheet 3

Spaces of continuous functions

Exercise 1. *Let $F : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by $F(f) := \int_0^1 f(x)dx$, where we consider the Riemann integral. Prove that F is continuous w.r.t. the uniform metric: i.e. show that for any $\varepsilon > 0$, we can find $\delta > 0$ such that if $\|f-g\|_\infty < \delta$, then $|F(f) - F(g)| < \varepsilon$. What does it say if f denotes the density of a line-like object?*

Proof. As in the statement, let $\varepsilon > 0$ and $f, g \in C(D,\mathbb{R})$ be such that $\|f-g\|_\infty \leq \delta$ with δ to be determined. We compute:

$$|F(f) - F(g)| = \left|\int_0^1 (f(x) - g(x))\mathrm{d}x\right| \leq \int_0^1 |f(x) - g(x)|\mathrm{d}x \leq \int_0^1 \|f-g\|_\infty \mathrm{d}x \leq \delta.$$

It follows from choosing $\delta = \varepsilon$ that F is continuous as a map $(C(D,\mathbb{R}), \|\cdot\|_\infty) \to (\mathbb{R}, |\cdot|)$. Note that changing the bounds of the integral from $[0,1]$ to $[a,b]$ leads to choosing $\delta = \varepsilon/(b-a)$.

This means that computing the mass of a line-like object given its density is a continuous operation: a slight change in the density (w.r.t. the norm $\|\cdot\|_\infty$) only results in a slight change of the mass. □

Exercise 2. *Show that the set of functions $f_n : x \mapsto \sin(nx), x \in [0,1]$ defined for all $n \geq 1$ admits no subsequence that converges w.r.t. the norm $|\cdot|_\infty$.*

Proof. Assume by contradiction that some subsequence $(f_{n_k})_{k\in\mathbb{N}}$ converges uniformly to some f, which is therefore continuous (as a uniform limit of continuous functions). Notice that for each $n \in \mathbb{N}$, $f_n(\pi/n) = 1$. On the other hand, for all large enough k, $\|f - f_{n_k}\|_\infty \leq 1/2$. Therefore, for all large enough k, $f(\pi/n_k) \geq 1/2$. Taking $k \to +\infty$, since f is continuous, we get $f(0) \geq 1/2$. But since $f(0) = \lim_{k\to+\infty} f_{n_k}(0) = 0$, we get a contradiction. □

Fourier

Exercise 3. *Prove that the following orthogonality relations hold for integers $m, n \geq 0$:*

1. *Cosine-cosine Orthogonality:*

$$\int_0^1 \cos(2\pi nx)\cos(2\pi mx)\,dx = \begin{cases} 1, & \text{if } n = m = 0, \\ \frac{1}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

2. *Sine-sine Orthogonality:*

$$\int_0^1 \sin(2\pi nx)\sin(2\pi mx)\,dx = \begin{cases} 0, & \text{if } n = 0 \text{ or } m = 0, \\ \frac{1}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

3. *Sine-cosine Orthogonality:*

$$\int_0^1 \sin(2\pi nx)\cos(2\pi mx)\,dx = 0 \quad \forall n, m.$$

Proof. We compute

$$\langle \cos(2\pi n\cdot), \cos(2\pi m\cdot)\rangle := \int_0^1 \cos(2\pi nx)\cos(2\pi mx)\, dx.$$

Recall that

$$\cos\theta\cos\phi = \frac{1}{2}\left(\cos(\theta-\phi)+\cos(\theta+\phi)\right),$$

we rewrite first expression as

$$\langle \cos(2\pi n\cdot), \cos(2\pi m\cdot)\rangle = \int_0^1 \frac{1}{2}\left(\cos(2\pi(n-m)x)+\cos(2\pi(n+m)x)\right) dx.$$

Using the linearity of the integral, we can evaluate each summand in the integral separately using:

$$\int_0^1 \cos(2\pi kx)dx = \begin{cases} 1, & k=0, \\ 0, & k\neq 0. \end{cases}$$

Thus,

$$\langle \cos(2\pi n\cdot), \cos(2\pi m\cdot)\rangle = \frac{1}{2}\left(\delta_{n,m}+\delta_{n,-m}\right).$$

Since for nonzero n, $\delta_{n,-m}=0$, we get:

$$\langle \cos(2\pi n\cdot), \cos(2\pi m\cdot)\rangle = \begin{cases} 1, & \text{if } n=m=0, \\ \frac{1}{2}, & \text{if } n=m\neq 0, \\ 0, & \text{if } n\neq m. \end{cases}$$

Similarly,

$$\langle \sin(2\pi n\cdot), \sin(2\pi m\cdot)\rangle = \int_0^1 \sin(2\pi nx)\sin(2\pi mx)\, dx$$

can be computed using

$$\sin\theta\sin\phi = \frac{1}{2}\left(\cos(\theta-\phi)-\cos(\theta+\phi)\right)$$

which yields

$$\langle \sin(2\pi n\cdot), \sin(2\pi m\cdot)\rangle = \begin{cases} 0, & \text{if } n=0 \text{ or } m=0, \\ \frac{1}{2}, & \text{if } n=m\neq 0, \\ 0, & \text{if } n\neq m. \end{cases}$$

Finally, we rewrite

$$\langle \sin(2\pi n\cdot), \cos(2\pi m\cdot)\rangle = \int_0^1 \sin(2\pi nx)\cos(2\pi mx)\, dx$$

using the identity

$$\sin\theta\cos\phi = \frac{1}{2}\left(\sin(\theta+\phi)+\sin(\theta-\phi)\right).$$

This yields

$$\langle \sin(2\pi n\cdot), \cos(2\pi m\cdot)\rangle \int_0^1 \frac{1}{2}\left(\sin(2\pi(n+m)x)+\sin(2\pi(n-m)x)\right) dx = 0$$

as the integral of a sine function over a full period is always zero.

□

Remark: Do not worry if you do not know the trigonometric identities we used above, as you can always show them quite easily from scratch. Indeed, using the functional equation of the exponential together with Euler's identity, one gets

$$\cos(\theta+\phi)+i\sin(\theta+\phi)=\exp(i(\theta+\phi))=\exp(i\theta)\exp(i\phi)=(\cos(\theta)+i\sin(\theta))(\cos(\phi)+i\sin(\phi)).$$

Expanding the product on the right hand side and comparing real and imaginary parts on the two ends of the above equations yields the necessary ingredients (it is worth doing it once if you have not done so already).

Exercise 4. *Suppose that $f \in C([0,1],\mathbb{R})$ is k times continuously differentiable and satisfies $f^j(0) = f^j(1)$ for all $j = 0\dots k-1$* [1] *Then prove that there is some $C > 0$ such that for all $n \geq 1$ $|c_n| \leq Cn^{-k}$ and $|s_n| \leq Cn^{-k}$.*

Proof. Consider such f and and $1 \leq j \leq k$. Note that $f^{(j)}$ is continuous on the compact set $K := [0,1]$. Its absolute value therefore attains its maximum over K and thus $f^{(j)}$ is integrable:

$$\int_K |f^{(j)}|d\lambda(x) \leq ||f^{(j)}||_{K,\infty} =: C < \infty.$$

Hence, in particular, since $|\sin(\theta)|, |\cos(\theta)| \leq 1$ for $\theta \in \mathbb{R}$, its Fourier coefficients are well-defined and

$$|c_n(f^{(j)})(n)| \leq \int_K |f^{(j)}(x)\cos(2\pi nx)|\mathrm{d}\lambda(x) \leq C$$

as well as

$$|s_n(f^{(j)})(n)| \leq \int_K |f^{(j)}(x)\sin(2\pi nx)|\mathrm{d}\lambda(x) \leq C.$$

On the other hand, we compute

$$\begin{aligned}
|c_n(f^{(j)})(n)| &= \left|\int_K f^{(j)}(x)\cos(2\pi nx)\mathrm{d}\lambda(x)\right| \\
&= \left|\int_K (2\pi n) f^{(j-1)}(x)\sin(2\pi nx)\mathrm{d}\lambda(x)\right| = ... \\
&= \left|\int_K (2\pi n)^j f(x)\frac{d^j}{dx^j}\cos(u)|_{u=2\pi nx}\mathrm{d}\lambda(x)\right| = (2\pi n)^j\left|\int_K f(x)\frac{d^j}{dx^j}\cos(u)|_{u=2\pi nx}\mathrm{d}\lambda(x)\right|
\end{aligned}$$

by integrating by parts (notice that the boundary terms vanish due to the assumption that $f^j(0) = f^j(1)$). Moreover, the derivative in the integrand is, up to a sign, either a cosine or a sine. Therefore, the integral is equal to either the cosine or the sine coefficient of f, if j is even or odd, respectively. Together with the upper bounds on both the cosine and sine coefficients of the derivatives of f, one obtains for $j = k$:

$$C \geq |c_n(f)|n^k$$

and

$$C \geq |s_n(f)|n^k,$$

thereby concluding our proof. □

Remark 1: The condition $f^j(0) = f^j(1)$ for all $j = 0\dots k-1$ can be viewed as imposing certain periodicity conditions on f and its derivatives by identifying $0 \sim 1$, that is, K can be seen as the unit circle $\mathbb{S}_1$ and f as a $\mathcal{C}^k$-function on it.

Remark 2: The regularity of the function f therefore plays a role in the decay of the sequences of Fourier coefficients as $n \to \infty$.

[1] Here by $f^j(x)$ we mean the $j-th$ derivative of f at x, the $0-$th derivative being the function itself.