# Exercise sheet 14

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

## More Fourier series and transforms

**Exercise 1** (Legendre polynomials). *The goal of this exercise is to show that the Gram-Schmidt orthogonalization procedure applied to the sequence of monomials $(x \mapsto x^n)_{n\geq 0}$ on $L^2([-1,1])$ yields the (normalized) Legendre polynomials $(\sqrt{n+1/2}P_n)_{n\geq 0}$, where*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad P_0 = 1, P_1(x) = x.$$

1. *Show that at the $n$-th step of Gram-Schmidt, the projection of the monomial $x^n$ onto the span of $\{p_k\}_{k=1}^{n-1}$ is the same as that of $xp_{n-1}$. Then, use it to simplify the expression of $p_n$ given by Gram-Schmidt.*
2. *Further simplify the expression by showing by induction that $p_n$ is an even function for $n$ even, and an odd function when $n$ is odd.*
3. *Conclude that $p_n = P_n \cdot p_n(1)$, by showing that they satisfy the same recurrence relation.*

*Proof.* 1. By applying the Gram-Schmidt orthogonalization procedure, we get the recursion

$$p_n(x) = x^n - \sum_{k=1}^{n-1} \frac{\langle p_k, x^n\rangle}{\langle p_k, p_k\rangle} p_k(x) \tag{1}$$

We can simplify the right side of the equation in the following way. Note that $xp_{n-1}(x)$ is of degree $n$ and has leading coefficient 1. Hence, instead of projecting $x^n$ we can project $xp_{n-1}(x)$ and we get the same result. More formally, $xp_{n-1}(x) = x^n + q(x)$, where $q(x)$ is a polynomial of degree $n-1$. In particular $q(x) \in \text{span}(p_1, \dots, p_{n-1})$. This implies

$$q(x) = \sum_{k=1}^{n-1} \frac{\langle q, p_k\rangle}{\langle p_k, p_k\rangle} p_k(x).$$

With this we can rewrite equation 1 as

$$\begin{aligned} p_n(x) &= x^n - \sum_{k=1}^{n-1} \frac{\langle p_k, x^n\rangle}{\langle p_k, p_k\rangle} p_k(x) + q(x) - \sum_{k=1}^{n-1} \frac{\langle q, p_k\rangle}{\langle p_k, p_k\rangle} p_k(x) \\ &= x^n + q(x) - \sum_{k=1}^{n-1} \frac{\langle p_k, x^n + q(x)\rangle}{\langle p_k, p_k\rangle} p_k(x) \\ &= xp_{n-1}(x) - \sum_{k=1}^{n-1} \frac{\langle p_k, xp_{n-1}\rangle}{\langle p_k, p_k\rangle} p_k(x) \end{aligned}$$

Now note that $\langle p_k, xp_{n-1}\rangle = \langle xp_k, p_{n-1}\rangle$ and $xp_k$ is of degree $k+1$, hence $xp_k \in \text{span}(p_1, \dots, p_{k+1})$. Since $p_{n-1}$ is orthogonal to $\text{span}(p_1, \dots, p_{k+1})$ for $k+1 < n-1$, we conclude that $\langle xp_k, p_{n-1}\rangle = 0$ for $k+1 < n-1$. This gives

$$p_n(x) = xp_{n-1}(x) - \frac{\langle p_{n-1}, xp_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle} p_{n-1}(x) - \frac{\langle p_{n-2}, xp_{n-1}\rangle}{\langle p_{n-2}, p_{n-2}\rangle} p_{n-2}(x).$$

2. We show the statement by induction on $n$. For $n = 0$ and $n = 1$ the statement clearly holds. Consider now $n \geq 1$, and let us assume it also even, as the odd case works analogously. Then by the induction hypothesis, $p_{n-1}$ is odd. In particular $x(p_{n-1}(x))^2$ is also odd and this implies that $\langle p_{n-1}, xp_{n-1}\rangle = 0$. We have

$$\begin{aligned} p_n(-x) &= -xp_{n-1}(-x) - \frac{\langle p_{n-2}, xp_{n-1}\rangle}{\langle p_{n-2}, p_{n-2}\rangle} p_{n-2}(-x) \\ &= xp_{n-1}(x) - \frac{\langle p_{n-2}, xp_{n-1}\rangle}{\langle p_{n-2}, p_{n-2}\rangle} p_{n-2}(x) = p_n(x). \end{aligned}$$

3. We will have that $p_n = P_n \cdot p_n(1)$, where $P_n$ is the Legendre polynomial, provided that we can prove that they satisfy the same recurrence relation. Denoting $s_n := p_n(1)$, we get that

$$xs_{n-1}P_{n-1}(x) - s_{n-1}\frac{\langle P_{n-2}, xP_{n-1}\rangle}{\langle P_{n-2}, P_{n-2}\rangle} P_{n-2}(x) = xs_{n-1}P_{n-1}(x) - s_{n-1}\frac{n-1}{2n-1} P_{n-2}(x),$$

where we have used that $nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$ and their orthogonality. To conclude, it suffices to observe (for instance by induction) that $s_n/s_{n-1} = n/(2n-1)$, so that the last term is equal to $s_nP_n$, proving the claim.

□

**Exercise 2.** *In this exercise we aim to show that $L^2(\mathbb{R})$ is separable.*

- *Prove that each $f \in L^2(\mathbb{R})$ can be approximated arbitrarily well by $f1_{[-n,n]}$ for n large enough, meaning that for every $\varepsilon > 0$ there is some $n \in \mathbb{N}$ with $\|f - f1_{[-n,n]}\|_2 < \varepsilon$.*
- *By using dense countable subsets of $L^2([-n, n])$ (which we know to be separable), find a countable dense subset of $L^2(\mathbb{R})$.*

[1].

*Proof.* • For $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$, the function $f^2\mathbf{1}_{-N,N}$ is integrable by monotonicity and its integral tends to that of $f^2$ by the dominated convergence theorem: therefore there exists $N \geq 1$ such that $\|f - f\mathbf{1}_{-N,N}\|_{L^2(\mathbb{R})} \leq \varepsilon/2$.

- Take $\{g_i^N\}_{i\geq 1}$ a countable dense subset of $L^2([-N, N])$ (for instance, all trigonometric polynomials with rational coefficients). We then claim that

$$\{g_i^N\,\mathbf{1}_{[-N,N]}\}_{i,N\geq 1}$$

is the desired set (these functions lie all in $L^2$). Indeed, for all $f \in L^2(\mathbb{R})$, by the first part there exists $N \geq 1$ s.t. by the triangular inequality $\|f - f\mathbf{1}_{-N,N}\|_{L^2(\mathbb{R})} \leq \varepsilon/2$, and by density there exists $N_k \in \mathbb{N}$ such that $\left\|f\mathbf{1}_{-N,N} - g_{N_k}^N\right\|_{L^2([-N,N])} \geq \varepsilon/2$, from which it follows that $\left\|f - g_{N_k}^N\mathbf{1}_{-N,N}\right\|_{L^2(\mathbb{R})} \leq \varepsilon$.

□

**Exercise 3** (Heisenberg uncertainty principle)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function such that all its derivatives decay more than polynomially fast - more precisely, such that $x^m f^{(n)}$ are bounded for any $m, n \in \mathbb{N}$ with $f^{(n)}$ denoting the $n-$th derivative (this is called a Schwartz function). Suppose further that $\int_{\mathbb{R}} |f|^2 d\lambda = 1$. Show that*

[1]Why is $\{\exp(2\pi ikx)\}_{k\in\mathbb{R}}$ not a basis?
Actually, a basis can be neatly constructed from Hermite polynomials and the Hamiltonian of the quantum harmonic oscillator, as studied in Exercise 8.

- *Also* $\int_{\mathbb{R}} |\hat{f}|^2 d\lambda = 1$*;*
- *both* $x^2|f(x)|^2$ *and* $k^2|\hat{f}(k)|^2$ *are integrable;*

*Hint: look at the Fourier transform of* $f'$*.*

- *the following uncertainty principle holds:*

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2 d\lambda(x)\right)\left(\int_{\mathbb{R}} k^2|\hat{f}(k)|^2 d\lambda(x)\right) \geq \frac{1}{16\pi^2}.$$

- *In fact for any* $x_0, k_0$*:*

$$\left(\int_{\mathbb{R}} (x_0 - x)^2|f(x)|^2 d\lambda(x)\right)\left(\int_{\mathbb{R}} (k_0 - k)^2|\hat{f}(k)|^2 d\lambda(x)\right) \geq \frac{1}{16\pi^2}.$$

*Intuitively this says that for any function both* $f$ *and* $\hat{f}$ *cannot be simultaneously localised. The interpretation in the realm of quantum mechanics is that the position of the particle and its momentum cannot be localised simultaneously.*

*Hint: look at the function* $x \mapsto \alpha x f + \beta f'(x)$ *and expand its* $L^2$ *norm.*

*Proof.*

- It follows directly from Plancherel's formula.
- We have that for all $x \in \mathbb{R}$,

$$\begin{aligned} x^2|f(x)|^2 &= \mathbf{1}_{|x|<1}x^2|f(x)|^2 + \mathbf{1}_{|x|\geq 1}x^2|f(x)|^2 \\ &\leq \sup_{y\in\mathbb{R}} |f(y)|^2\mathbf{1}_{|x|<1} + \sup_{y\in\mathbb{R}} y^4|f(y)|^2\mathbf{1}_{|x|\geq 1}\frac{1}{x^2}, \end{aligned}$$

  which we have seen is integrable (the suprema are finite by assumption). We conclude by monotonicity.

  For the Fourier Transform, we first notice that $f' \in L^1$, since for all $x \in \mathbb{R}$,

$$|f'(x)| \leq (1 + x^2)|f'(x)|\frac{1}{1 + x^2} \leq \sup_{y\in\mathbb{R}}(|f'(y)|(1 + y^2))\frac{1}{1 + x^2}$$

  which is integrable again since the supremum is finite. We can therefore consider its Fourier transform:

$$\widehat{f'}(k) = \int_{\mathbb{R}} f'(x)e^{-2\pi ikx}\mathrm{d}\lambda(x) = 2\pi ik\int_{\mathbb{R}} f(x)e^{-2\pi ikx}\mathrm{d}\lambda(x) = 2\pi ik\hat{f}(k),$$

  where the integration by parts is justified like in Exercise 3, exercise sheet 13:

  We then have that for $k \in \mathbb{R}$,

$$k^2|\hat{f}(k)|^2 = \frac{1}{4\pi^2}|\widehat{f'}(k)|^2,$$

  and since $f' \in L^2$ using similar arguments as for $x^2|f(x)|^2$ (by noting that $f'$ is also a Schwartz function) we conclude by Plancherel's formula that the right-hand term lies in $L^2$ with norm equal to $\|f'\|_{L^2}/(4\pi^2)$.

- For any $a, b \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} (axf(x) + bf'(x))(\overline{axf(x) + bf'(x)})d\lambda(x) \\ &= a^2\int_{\mathbb{R}} x^2|f(x)|^2 d\lambda(x) + b^2\int_{\mathbb{R}} |f'(x)|^2 d\lambda(x) + ab\int_{\mathbb{R}} xf(x)\overline{f'(x)} + xf'(x)\overline{f(x)}d\lambda(x) \\ &= a^2\int_{\mathbb{R}} x^2|f(x)|^2 d\lambda(x) + b^2\int_{\mathbb{R}} |f'(x)|^2 d\lambda(x) - 2ab\int_{\mathbb{R}} |f(x)|^2 d\lambda(x), \end{aligned}$$

where we used integration by parts in the third integral, together with the fact that $xf(x)$ goes to zero for $x \to \pm\infty$, since $f$ decays more than polynomially fast. Now we set $a^2 = \int_{\mathbb{R}} |f'(x)|^2 d\lambda(x)$ and $b^2 = \int_{\mathbb{R}} x^2 |f(x)|^2 d\lambda(x)$ and divide both sides of the inequality by $ab$ to get

$$ab \geq \frac{1}{2\int_{\mathbb{R}} |f(x)|^2 d\lambda(x)} = \frac{1}{2}. \tag{2}$$

Finally we can write $a^2$ differently by using as we've seen that $\widehat{f'}(k) = 2\pi i k \widehat{f}(k)$ and using the Plancherel formula we get

$$a^2 = \int_{\mathbb{R}} |f'(x)|^2 d\lambda(x) = \int_{\mathbb{R}} |\widehat{f'}(k)|^2 d\lambda(k) = 4\pi^2 \int_{\mathbb{R}} k^2 |\hat{f}(k)|^2 d\lambda(k)$$

Plugging this into equation (2)and rearranging, we conclude that

$$\int_{\mathbb{R}} x^2 |f(x)|^2 \int_{\mathbb{R}} k^2 |\hat{f}(k)|^2 d\lambda(k) \geq \frac{1}{16\pi^2}.$$

For the last bullet, note that

$$\begin{aligned} f(\widehat{x+x_0})(k) &= \int_{\mathbb{R}} \exp(-2\pi i k x) f(x+x_0) d\lambda(x) = \exp(2\pi i k x_0) \int_{\mathbb{R}} \exp(-2\pi i k (x+x_0)) f(x+x_0) d\lambda(x) \\ &= \exp(2\pi i k x_0) \hat{f}(k). \end{aligned}$$

Let us define $g(x) = \exp(-2\pi i x k_0) f(x+x_0)$. We have

$$\int_{\mathbb{R}} x^2 |g(x)|^2 d\lambda(x) = \int_{\mathbb{R}} x^2 |f(x+x_0)|^2 d\lambda(x) = \int_{\mathbb{R}} (x-x_0)^2 |f(x)|^2 d\lambda(x) \tag{3}$$

On the other hand, if we compute the Fourier transform of $g$ we get

$$\begin{aligned} \hat{g}(k) &= \int_{\mathbb{R}} \exp(-2\pi i x k) \exp(-2\pi i x k_0) f(x+x_0) d\lambda(x) \\ &= \int_{\mathbb{R}} \exp(-2\pi i x (k+k_0)) f(x+x_0) d\lambda(x) \\ &= f(\widehat{x+x_0})(k+k_0) \end{aligned} \tag{4}$$

This implies that

$$\begin{aligned} \int_{\mathbb{R}} k^2 |\hat{g}(k)|^2 d\lambda(k) &= \int_{\mathbb{R}} k^2 |f(\widehat{x+x_0})(k+k_0)|^2 d\lambda(k) \\ &= \int_{\mathbb{R}} (k-k_0)^2 |f(\widehat{x+x_0})(k)|^2 d\lambda(k) \\ &= \int_{\mathbb{R}} (k-k_0)^2 |\hat{f}(k)|^2 d\lambda(k), \end{aligned}$$

where in the last equation we used equation (4). By applying the previously derived uncertainty principle to the function $g$, we conclude the proof of the last part of the exercise.

□

## Operators

**Exercise 4.** *Consider $(\mathcal{H}, \|\cdot\|)$ a separable Hilbert space and $T$ a bounded linear operator.*

- *Show the following inequality for $f \in H$:*

$$\|Tf\| \leq \|T\|_{op}\|f\|.$$

- *Show that $T$ is continuous in the sense that if a sequence $(f_n)_{n\geq 1}$ converges to $f$ w.r.t $\|\cdot\|$, then also $Tf_n$ converges to $Tf$.*

*Proof.*

- Let $f \in \mathcal{H}$:

$$\|Tf\| = \|f\| \cdot \left\| T\frac{f}{\|f\|} \right\| \leq \|f\| \sup_{g\in\mathcal{H},\|g\|\leq 1} \|Tg\| = \|f\| \cdot \|T\|_{\text{op}}.$$

- Consider $(f_n)_{n\geq 1} \subset \mathcal{H}$ converging to some $f \in \mathcal{H}$. We have that

$$\|Tf_n - Tf\| \leq \|f_n - f\|\|T\|_{\text{op}} \xrightarrow[n\to\infty]{} 0,$$

therefore $(Tf_n)_{n\geq 1}$ converges to $Tf$ as desired.

□

**Exercise 5** (Boundedness of operators)**.** *We aim to argue that the position operator formally given by $f \to xf$ is not bounded on $L^2(\mathbb{R})$:*

- *Find a square-integrable function $f$ such that $xf(x)$ is not square integrable.*
- *Show that the position operator is well defined for functions $f$ that are square integrable and such that also $xf$ is square integrable.*
- *Find for $i \geq 1$ functions $f_i \in L^2(\mathbb{R})$ of unit norm with $xf_i \in L^2(\mathbb{R})$ but $\|xf_i\|_2 \to \infty$.*

*Proof.* For the first bullet point, consider the function $f(x) = \frac{1}{x}1_{[1,\infty)}$. We have $\|f\|_2 = 1$ but $\|xf\|_2 = \left\|1_{[1,\infty)}\right\|_2 = \infty$. For the second bullet point, it is clear that the assumptions on $f$ imply that the operator is finite. For the third bullet point consider the functions $f_i = 1_{[i,i+1]}$. We have $\|f_i\|_2 = 1$, but

$$\|xf_i\|_2^2 = \int_i^{i+1} x^2 dx = \frac{1}{3}\big((i+1)^3 - i^3\big) = i^2 + i + \frac{1}{3},$$

which goes to infinity as $i \to \infty$. □

**Exercise 6** (Finite rank linear operators)**.** *Consider a real Hilbert space $\mathcal{H}$. Let $u_1, \ldots, u_n \in \mathcal{H}$, $v_1, \ldots, v_n \in \mathcal{H}$, and define $T(f) := \sum_{i=1}^n \langle f, u_i\rangle v_i$.*

- *Prove that $T$ is bounded.*

*Now let $T$ be of finite rank and Hermitian, i.e. $\langle Tf, g\rangle = \langle f, Tg\rangle$ for any $f, g \in \mathcal{H}$.*

- *Argue that $T$ can be diagonalized, i.e. that we can find $g_1, \ldots g_m$ orthonormal with $m \leq n$ and $\lambda_i \in \mathbb{R}$ such that if we write $f = \sum_{i=1}^m c_i\langle f, g_i\rangle g_i + f_0$ with $f_0$ orthogonal to $g_1, \ldots, g_m$ then*

$$T(f) = \sum_{i=1}^m c_i\lambda_i\langle f, g_i\rangle g_i.$$

*Proof.*

- For the first bullet point, we compute

$$\begin{aligned}\|T(f)\| &= \left\|\sum_{i=1}^{n}\langle f, u_i\rangle v_i\right\| \\ &\leq \sum_{i=1}^{n} |\langle f, u_i\rangle|\|v_i\| \\ &\leq \|f\| \sum_{i=1}^{n} \|u_i\|\|v_i\|,\end{aligned}$$

  where in the last inequality we used Cauchy-Schwarz. We deduce that

$$\sup_{\|f\|\leq 1} \|T(f)\| \leq \sum_{i=1}^{n} \|u_i\|\|v_i\| < \infty.$$

- First, notice that $V = \text{span}(u_1, \ldots, u_n, v_1, \ldots, v_n)$ is a subspace of $\mathcal{H}$ and we can extract an orthonormal basis $\{h_1, \ldots, h_m\}$ for some $1 \leq m \leq 2n$ (so that $V$ has dimension $m$). In particular, check that we may rewrite

$$T = \sum_{i,j=1}^{m} \beta_{ij}\langle\cdot, h_i\rangle h_j.$$

  $T|_V$ is then a linear map from $V$ to $V$, a finite dimensional space: using the spectral theorem from linear algebra, we can diagonalize $T|_V$, i.e. find another orthonormal basis $\{g_1, \cdots, g_m\}$ of $V$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ (the eigenvalues are real as $T$ is Hermitian; and in fact, at most $n$ of these eigenvalues are non-zero since $\text{rank}(T) \leq n$, so we can now assume w.l.o.g. that $m \leq n$) such that

$$T|_V(\sum_{i=1}^{m} \alpha_i g_i) = \sum_{i=1}^{m} \alpha_i\lambda_i g_i$$

  for all $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Now we recall that we can write $\mathcal{H} = V \oplus V^{\perp}$, so that any $f \in \mathcal{H}$ can be written $f = f_0 + f^0$, $f_0 \in V^{\perp}$, $f^0 \in V$. Furthermore, there exist $c_1, \ldots, c_m \in \mathbb{R}$ such that $f^0 = \sum_{i=1}^{m} c_i \left\langle f^0, g_i\right\rangle g_i = \sum_{i=1}^{m} c_i \langle f, g_i\rangle g_i$. We can finally conclude:

$$T(f) = T(f_0 + f^0) = T(f_0) + T|_V \left(\sum_{i=1}^{m} c_i \langle f, g_i\rangle g_i\right) = \sum_{i=1}^{m} c_i\lambda_i \langle f, g_i\rangle g_i$$

□