Exercise sheet 13

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1 (Fourier series on larger intervals)**.** *Using Fourier series on* $[0,1]$ *and scaling / translation show that for any* $L \in \mathbb{N}$*, every* $f_L \in L^2([-L/2, L/2])$ *can be written as*

$$f_L(x) = L^{-1} \sum_{n\in\mathbb{Z}} \hat{f}_L(n/L) \exp\big(2\pi i L^{-1} n x\big), \tag{1}$$

where the summing is absolute for any $x \in [-L/2, L/2]$ *and the limit in the series is with respect to the* L^2 *norm, and*

$$\hat{f}_L(n/L) := \int_{[-L/2,L/2]} f_L(x) \exp\big(-2\pi i L^{-1} n x\big) d\lambda(x). \tag{2}$$

Deduce Lemma 3.22, i.e. that the set of functions: $(\sqrt{\frac{2}{L}}\sin(\frac{2}{L}\pi n x))_{n\geq 1}, (\sqrt{\frac{2}{L}}\cos(\frac{2}{L}\pi n x))_{n\geq 1}$ *together with the constant function* $\frac{1}{\sqrt{L}}$ *forms an orthonormal basis of* $L^2([-L/2, L/2))$.

Proof. Define the function $F(x) = f_L((x-1/2)L)$. Then $F \in L^2([0,1])$ and by completeness of the Fourier basis $(\exp(2\pi i n\cdot))_{n\in\mathbb{Z}}$ (alternatively, of $(\sin(2\pi n\cdot))_{n\geq 1}, (\cos(2\pi n\cdot))_{n\geq 1}, 1)$ there are unique coefficients $(\hat{F}_n)_{n\geq 1} \subset \mathbb{R}$ such that F can be written

$$F(x) = \sum_{n\in\mathbb{Z}} \hat{F}(n) e^{2\pi i n x}. \tag{3}$$

We compute more specifically

$$\begin{aligned}
\hat{F}(n) &= \int_{\mathbb{R}} F(x) 1_{[0,1]}(x) \exp(-2\pi i x n) d\lambda(x) \\
&= \int_{\mathbb{R}} F(x) 1_{[-L/2,L/2]}(L(x-1/2)) \exp(-2\pi i x n) d\lambda(x) \\
&= \int_{\mathbb{R}} f_L(L(x-1/2)) 1_{[-L/2,L/2]}(L(x-1/2)) \exp(-2\pi i x n) d\lambda(x) \\
&= \frac{(-1)^n}{L} \int_{\mathbb{R}} f_L(x) 1_{[-L/2,L/2]}(x) \exp\big(-2\pi i x n L^{-1}\big) d\lambda(x),
\end{aligned}$$

where in the last step we used the translation and dilation formula from exercise 1 of exercise sheet 11. We conclude that

$$\begin{aligned}
f_L(x) = F((x+1/2)/L) &= \sum_{n\in\mathbb{Z}} (-1)^n \hat{F}(n) \exp(2\pi i n x/L) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{L} \int_{[-L/2,L/2]} f_L(y) \exp(-2\pi i y n/L) d\lambda(y) \exp(2\pi i n x/L) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{L} \hat{f}(n/L) \exp(2\pi i n x/L).
\end{aligned}$$

□

Exercise 2 (Finishing Theorem 3.23)**.**

1. *In the set-up of Theorem 3.23, prove rigorously that the three following functions*

$$u_P(t,x) = \langle u_0, 1\rangle + \sum_{n\geq 1} \exp\bigl(-D4\pi^2 n^2 t\bigr)\left(\sin(2\pi n x)\langle u_0, 2\sin(2\pi n\cdot)\rangle + \cos(2\pi n x)\langle u_0, 2\cos(2\pi n\cdot)\rangle\right)$$
$$u_D(t,x) = \sum_{n\geq 1} \exp\bigl(-D\pi^2 n^2 t\bigr)\sin(\pi n x)\langle u_0, 2\sin(\pi n\cdot)\rangle$$
$$u_N(t,x) = \langle u_0, 1\rangle + \sum_{n\geq 1} \exp\bigl(-D\pi^2 n^2 t\bigr)\cos(\pi n x)\langle u_0, 2\cos(2\pi n\cdot)\rangle$$

are well-defined and belong to $L^2([0,1])$. *Furthermore, prove that for each* $t > 0$ *they are differentiable in* t *and twice differentiable in* x *such that the derivatives are Riemann integrable. Finally, show that they solve the heat equation.*

2. *Finish the proof of uniqueness in Theorem 3.23 by verifying the steps in the formal calculation. In particular, when* $u(t,x)$ *is one of the three functions above argue the following.*

   - *By using the connection to the Riemann integral and known results in that case or otherwise show that for all* $t > 0$,

   $$\frac{\partial \|u(t,x)\|^2}{\partial t} = 2\int_{[0,1]} u(t,x)\frac{\partial u(t,x)}{\partial t} d\lambda(x)$$

   - *Similarly show that the integration by parts is allowed:*

   $$\int_{[0,1]} u(t,x)\frac{\partial^2 u(t,x)}{\partial x^2} d\lambda(x) = -\int_{[0,1]} \left(\frac{\partial u(t,x)}{\partial x}\right)^2 d\lambda(x).$$

Proof. 1. We focus on u_D, the other functions can be studied identically. By assumption, $u_0(0) = u_0(1) = 0$, so the expression given for $u_D(0,\cdot)$ corresponds to the Fourier series expansion of u_0, which converges in L^2 as $u_0 \in L^2$ by assumption. Furthermore, the function $f : \mathbb{R}^+ \times [0,1] \to \mathbb{R}$,

$$f(t,x) = \exp\bigl(-D\pi^2 n^2 t\bigr)\sin(\pi n x)\langle u_0, 2\sin(\pi n\cdot)\rangle$$

is clearly smooth in both variables. Since

$$\bigl|\partial_x^2 f(t,x)\bigr| = \bigl|-\pi^2 n^2 \exp\bigl(-D\pi^2 n^2 t\bigr)\sin(\pi n x)\langle u_0, 2\sin(\pi n\cdot)\rangle\bigr| \leq 2\pi^2 n^2 \exp\bigl(-D\pi^2 n^2 t\bigr)\|u_0\|_2,$$

it follows that $\sum_{n=1}^{+\infty} \partial_x^2 f_n(t,\cdot)$ converges pointwise absolutely to a bounded function (and actually the convergence is even uniform): indeed for all ε there exists $N \geq 1$ large enough such that for all $K_1, K_2 \geq M$,

$$\left\|\sum_{n=1}^{K_1} \bigl|\partial_x^2 f_n(t,\cdot)\bigr| - \sum_{n=1}^{K_2} \bigl|\partial_x^2 f_n(t,\cdot)\bigr|\right\|_\infty \leq \sum_{n=N}^{+\infty} 2\pi^2 n^2 \exp\bigl(-D\pi^2 n^2 t\bigr)\|u_0\|_2 \leq \varepsilon.$$

Arguing similarly for $\sum_{n=1}^{+\infty} f_n(t,\cdot)$ and $\sum_{n=1}^{+\infty} \partial_x f_n(t,\cdot)$, we obtain that $u_D(t,\cdot) = \sum_{n=1}^{+\infty} f_n(t,\cdot)$ is twice differentiable in x and that

$$\partial_x^2 u_D(t,x) = -\sum_{n\geq 1} \pi^2 n^2 \exp\bigl(-D\pi^2 n^2 t\bigr)\sin(\pi n x)\langle u_0, 2\sin(\pi n\cdot)\rangle.$$

The reasoning for $\partial_t f_n(x,\cdot)$, $x \in [0,1]$ fixed is the same[1]: we can apply the hint because the series of $\partial_t f_n(x,t)$ converges in the space of continuous functions, given that it is a bounded

[1] Here $t \in (0,+\infty)$ whereas in the hint we ask for the functions to be defined on $[0,1]$, but that can easily be circumvented by considering $t \in [n, n+1]$, i.e. $t - n \in [0,1]$.

series of continuous functions and $C^0([0,1])$ is complete with the uniform norm. We obtain therefore that $u_D(\cdot,x)$ is differentiable in t and that

$$\partial_t u_D(t,x) = -\sum_{n\geq 1} D\pi^2 n^2 \exp\bigl(-D\pi^2 n^2 t\bigr)\sin(\pi n x)\langle u_0, 2\sin(\pi n\cdot)\rangle.$$

In particular, it holds that $\partial_t u = D\Delta u$. Lastly, $u_D(t,\cdot)$ satisfies the Dirichlet condition for all $t>0$ as well.

For u_N and u_P, one argues similarly, with the only difference corresponding to the boundary conditions: for u_P, notice that $u_P(t,0) = u_P(t,1)$ for all $t>0$, and for u_N one must differentiate at the boundary of $[0,1]$. Since cos is periodic, we can actually see $u_N(t,\cdot)$ as a function defined on $[-1,2]$, and the derivative can be computed inside the sum using similar arguments to what has been done before. We obtain

$$\partial_x u_N(t,x) = -\sum_{n\geq 1} \pi n \exp\bigl(-D\pi^2 n^2 t\bigr)\cos(\pi n x)\langle u_0, 2\cos(\pi n\cdot)\rangle,$$

which gives the same value when evaluated at 0 or 1.

2. We have proven in the previous subquestion that $u_i, i\in\{P,D,N\}$ is differentiable once in t and twice in x (actually, notice that we could have proven that it was in fact infinitely differentiable!). In particular,

$$\|u(t,\cdot)\|^2_{L^2([0,1])} = \int_{[0,1]} u(t,x)^2 \mathrm{d}x$$

coincides with the Riemann integral since u is continuous, and one can use results from Analysis II on permutations of integrals and derivatives (since u is continuously differentiable in t) to obtain that

$$\frac{\partial\|u(t,\cdot)\|^2_{L^2([0,1])}}{\partial t} = \int_{[0,1]} 2u(t,x)\partial_t u(t,x)\mathrm{d}x.$$

Since u satisfies the heat equation, we may rewrite

$$\frac{\partial\|u(t,\cdot)\|^2_{L^2([0,1])}}{\partial t} = 2D\int_{[0,1]} u(t,x)\partial^2_{xx} u(t,x)\mathrm{d}x.$$

Lastly, the integrand is again a continuous function of x (on $[0,1]$, compact) so that we may integrate by parts (again because $u(t,\cdot)$ is twice differentiable) and conclude that

$$\frac{\partial\|u(t,\cdot)\|^2_{L^2([0,1])}}{\partial t} = -2D\int_{[0,1]} (\partial_x u(t,x))^2\mathrm{d}x.$$

□

Remark 1. *If u_0 is not continuous (think of $[0,1]$ as a rod of length 1, and imagine u_0 to be a step function, as if only a section of the rod had been heated), there is of course no hope that u_t converges uniformly as $t\to 0$. What is very surprising is that even for "rough/irregular" initial data u_0, we have seen that u_t becomes differentiable (actually, C^∞!) as soon as $t>0$. This regularization property of the heat equation is a very important idea in mathematics and physics.*

Exercise 3 (Fourier transform of Gaussian density)**.** *The aim of this exercise is to calculate the Fourier transform of the Gaussian density* $\exp\bigl(-x^2/2\bigr)$.

- *By allowing yourself to change the order of differentiation / integration and using integration by parts, find a first-order ODE satisfied by the Fourier transform*

$$\hat{f}(k) := \int_{\mathbb{R}} \exp\bigl(-x^2/2\bigr)\exp(-2ki\pi x)d\lambda(x).$$

- *Justify carefully the change of integration and differentiation and integration by parts in part 1.*
- *Solve this ODE and find thus the Fourier transform of the Gaussian density.*
- *Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is equal to its Fourier transform?*

Proof.

- Let $f(x) = \exp\bigl(-x^2/2\bigr)$. We compute first formally

$$\frac{d}{dk}\hat{f}(k) = \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right)(-2\pi i x)\exp(-2\pi i k x)d\lambda(x) = 2\pi i \int_{\mathbb{R}} \frac{d}{dx}\left(\exp\left(-\frac{x^2}{2}\right)\right)\exp(-2\pi i k x)d\lambda(x),$$

We formally apply integration by parts:

$$\frac{d}{dk}\hat{f}(k) = -4\pi^2 k \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right)\exp(-2\pi i k x)d\lambda(x) = -4\pi^2 k\hat{f}(k). \tag{4}$$

- To justify the differentiation, we first write for $k \in \mathbb{R}, h \neq 0$:

$$\frac{\hat{f}(k+h) - \hat{f}(k)}{h} = \int_{\mathbb{R}} e^{-x^2/2}\frac{e^{-2\pi i(k+h)x} - e^{-2\pi i k x}}{h}\mathrm{d}x.$$

We wish to take the limit $h \to 0$ by using the dominated convergence theorem. For any sequence $(h_n)_{n\geq 1} \subset \mathbb{R}^*$, let us define for $n \geq 1$ the function $f_n : \mathbb{R} \mapsto \mathbb{R}$,

$$f_n(x) = e^{-x^2/2}\frac{e^{-2\pi i(k+h_n)x} - e^{-2\pi i k x}}{h_n}.$$

These functions are continuous, and since

$$\left|\frac{e^{-2\pi i(k+h_n)x} - e^{-2\pi i k x}}{h_n}\right| \leq \left|\frac{1}{h_n}\int_k^{k+h_n} 2\pi e^{-2\pi i y x}\mathrm{d}y\right| \leq 2\pi,$$

the bound $|f_n(x)| \leq 2\exp\bigl(-x^2/2\bigr) =: g(x)$ implies that f_n is integrable and puts us in the set-up of the dominated convergence theorem. Since furthermore $(f_n)_{n\geq 1}$ converges pointwise to

$$\int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right)(-2\pi i x)\exp(-2\pi i k x)d\lambda(x),$$

the dominated convergence theorem implies that $\hat{f}$ is indeed differentiable and gives us the correct expression.

For the integration by parts, one should consider the integral on $[-K, K]$, i.e.

$$-4\pi^2 k \int_{-K}^{K} \exp\left(-\frac{x^2}{2}\right)\exp(-2\pi i k x)d\lambda(x),$$

such that it is the integral of a differentiable function on a compact interval and integration by parts can be performed, for instance because these results have been shown for the Riemann integral. It remains to take the limit $K \to \infty$, and this is possible by dominated convergence as the integrand is bounded by Cg which is in L^1.

- The ordinary differential equation (4) admits a unique solution given by

$$\hat{f}(k) = \hat{f}(0)\exp\bigl(-2\pi^2 k\bigr) = \sqrt{2\pi}\exp\bigl(-2\pi^2 k^2\bigr).$$

- Similarly, we can check that the Fourier transform of $h(x) = f(ax) = \exp\bigl(-a^2x^2/2\bigr)$ is given by

$$\hat{h}(k) = \frac{1}{a}\hat{f}(k/a) = \frac{\sqrt{2\pi}}{a}\exp\bigl(-2\pi^2 k^2/a^2\bigr).$$

  We want to find a such that h is its own Fourier transform, i.e.,

$$\frac{\sqrt{2\pi}}{a}\exp\bigl(-2\pi^2k^2/a^2\bigr) = \exp\bigl(-a^2k^2/2\bigr),$$

  which is satisfied for $a = \sqrt{2\pi}$. We conclude that $h(x) = \exp\bigl(-\pi x^2\bigr)$ equal to its own Fourier transform.

$\square$

Exercise 4 (Convolutions, II)**.** *Recall for g a bounded measurable function on $\mathbb{R}$ the convolution product $f \star g$ on $L^1(\mathbb{R})$ (i.e. for $f \in L^1$), defined by*

$$(f \star g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y.$$

Suppose now that g is also integrable, i.e. that $g \in L^1(\mathbb{R})$. Show that the following identity holds:

$$\mathcal{F}(f \star g) = \mathcal{F}(f) \cdot \mathcal{F}(g),$$

where $\cdot$ stands for pointwise multiplication.

Proof. We have already shown that $f \star g \in L^1$. We can compute its Fourier coefficients:

$$\mathcal{F}(f \star g)(k) = \int_{\mathbb{R}} (f \star g)(x)\exp(2\pi ikx)\mathrm{d}x = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y\right)\exp(2\pi ikx)\mathrm{d}x.$$

Since the integrand is bounded by $|f(x-y)g(y)\exp(2\pi ikx)| \le |f(x-y)g(y)|$ which is integrable as a function on $\mathbb{R}^2$, we can use Fubini's theorem to exchange the integrals. We obtain

$$\begin{aligned}
\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x-y)\exp(2\pi ikx)\mathrm{d}x\right)g(y)\mathrm{d}y &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x-y)\exp(2\pi ik(x-y))\mathrm{d}x\right)g(y)\exp(2\pi iky)\mathrm{d}y \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x)\exp(2\pi ikx)\mathrm{d}x\right)g(y)\exp(2\pi iky)\mathrm{d}y \\
&= \int_{\mathbb{R}} \mathcal{F}f(k)g(y)\exp(2\pi iky)\mathrm{d}y = \mathcal{F}f(k)\cdot\mathcal{F}g(k),
\end{aligned}$$

where we have used translation invariance of the Lebesgue measure on the second line. $\square$