Exercise sheet 12

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1 (Markov inequality). *Let f be integrable and non-negative and let $\beta > 0$. Prove that $\lambda(\{x : f(x) > \beta\}) \leq (\int f d\lambda)/\beta$.*

Conclude that if $f, g \in L^1(E)$ satisfy $\|f - g\|_1 \leq \varepsilon$, then $\lambda(\{x : |f(x) - g(x)| > \lambda\}) \leq \varepsilon/\lambda$.

Proof. For the first part, by monotonicity of the integral:

$$\lambda(\{x : f(x) > \beta\}) = \int \mathbf{1}_{\{x:f(x)>\beta\}} \mathrm{d}\lambda \leq \int \frac{f}{\beta} \mathbf{1}_{\{x:f(x)>\beta\}} \mathrm{d}\lambda \leq \frac{1}{\beta} \int f \mathrm{d}\lambda.$$

The second part follows immediately from applying Markov's inequality to $|f - g|$. □

Exercise 2. *Consider the form $\langle \cdot, \cdot \rangle$ on $L^2([0, 1], \mathbb{C})$ defined for $f, g \in L^2([0, 1], \mathbb{C})$ by*

$$\langle f, g \rangle = \int_{[0,1]} f \, \overline{g} \, \mathrm{d}\lambda,$$

is a (hermitian / complex) scalar product[1]*, turning $L^2([0, 1], \mathbb{C})$ into a complex vector space. Then, similarly to Exercise 3, sheet 3, show that $(\exp(2\pi i n \cdot))_{n\in\mathbb{Z}}$ are orthonormal functions in $L^2([0, 1])$.*

Proof. The form $\langle \cdot, \cdot \rangle$ is clearly sesquilinear by the linearity properties of the integral. Also, for any $f \in L^2([0, 1], \mathbb{C})$, $\langle f, f \rangle$ is real and positive since $f\overline{f} = |f|^2$ is. Finally, for $f, g \in L^2([0, 1], \mathbb{C})$, we check that

$$\begin{aligned}
\langle f, g \rangle &= \int_{[0,1]} f \, \overline{g} \, \mathrm{d}\lambda = \int_{[0,1]} \mathrm{Re}(f \, \overline{g}) \, \mathrm{d}\lambda + i \int_{[0,1]} \mathrm{Im}(f \, \overline{g}) \, \mathrm{d}\lambda \\
&= \int_{[0,1]} \mathrm{Re}(\overline{f} \, g) \, \mathrm{d}\lambda - i \int_{[0,1]} \mathrm{Im}(\overline{f} \, g) \, \mathrm{d}\lambda \\
&= \overline{\int_{[0,1]} g \, \overline{f} \, \mathrm{d}\lambda} = \overline{\langle g, f \rangle},
\end{aligned}$$

where we used the definition of the integral of a complex-valued function and linearity.

For orthogonality now, since both the real and the imaginary part of $(\exp(2\pi i n \cdot))_{n\in\mathbb{Z}}$ are continuous on $[0, 1]$, the Lebesgue and Riemann integrals coincide and we can use standard integration techniques. We need to integrate $\exp(2\pi i n x)\overline{\exp(2\pi i m x)} = \exp(2\pi i n x) \exp(-2\pi i m x) = \exp(2\pi i (n - m)x)$. We have two cases, either $m = n$ and

$$\int_0^1 \exp(2\pi i n x)\overline{\exp(2\pi i m x)} \mathrm{d}\lambda(x) = 1,$$

or $n \neq m$ and

$$\begin{aligned}
\int_0^1 \exp(2\pi i n x)\overline{\exp(2\pi i m x)} \mathrm{d}\lambda(x) &= \int_0^1 \exp(2\pi i (n - m)x) \mathrm{d}\lambda(x) \\
&= \int_0^1 \cos(2\pi (n - m)x) \mathrm{d}\lambda(x) + i \int_0^1 \sin(2\pi (n - m)x) \mathrm{d}\lambda(x) \\
&= \frac{1}{2\pi (n - m)} \Big(\big[\sin(2\pi (n - m)x) \big]_0^1 + \big[-\cos(2\pi (n - m)x) \big]_0^1 \Big) = 0,
\end{aligned}$$

which shows that $(\exp(2\pi i n \cdot))_{n\in\mathbb{Z}}$ is an orthonormal sequence. □

[1]For positive-definiteness of the scalar product, one would need to identify functions in $L^2([0, 1], \mathbb{C})$ that are equal almost everywhere, like in the real case.

Exercise 3. *Show that, on an inner product space $(V, \langle \cdot, \cdot \rangle)$, the application $v \mapsto \sqrt{\langle v, v \rangle}$ always defines a norm.*

Proof. Define
$$\|v\| = \sqrt{\langle v, v \rangle}\,.$$
We check the three norm axioms in turn. Since $\langle v, v \rangle \geq 0$, we have $\|v\| \geq 0$. Moreover,

$$\|v\| = 0 \iff \langle v, v \rangle = 0 \iff v = 0.$$

For any scalar α,
$$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v \rangle} = \sqrt{\alpha^2 \langle v, v \rangle} = |\alpha|\,\|v\|.$$
Lastly, we prove the triangular inequality using Cauchy-Schwartz. For $v, w \in V$,

$$\|v + w\|^2 = \langle v + w, v + w \rangle = \|v\|^2 + 2\,\langle v, w \rangle + \|w\|^2 \leq \|v\|^2 + 2\,\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

Taking square roots gives
$$\|v + w\| \leq \|v\| + \|w\|.$$
Thus, all three norm axioms hold and $\|\cdot\|$ is indeed a norm on V. □

Exercise 4. *Let $v_1, v_2, \ldots$ be orthonormal vectors in a complete inner product space V. Show that for any $w \in V$, we have that $\widehat{w} := \sum_{i \geq 1} \langle v_i, w \rangle v_i$ is well-defined and satisfies 1) $\|\widehat{w}\| \leq \|w\|$ and 2) $\langle w - \widehat{w}, v_i \rangle = 0$ for all $i \geq 1$.*

Proof. Let $(v_i)_{i \geq 1}$ be orthonormal nonzero vectors in the inner product space V, and fix $w \in V$. For each $N \geq 1$ define the N-term projection

$$w_N \;=\; \sum_{i=1}^{N} \langle v_i, w \rangle\, v_i.$$

We can compute

$$\langle w - w_N, w_N \rangle = \sum_{i=1}^{N} \langle w, \langle v_i, w \rangle\, v_i \rangle - \sum_{j,k=1}^{N} \langle \langle v_j, w \rangle\, v_j, \langle v_k, w \rangle\, v_k \rangle = \sum_{j=1}^{N} |\langle v_j, w \rangle|^2 - \sum_{j=1}^{N} |\langle v_j, w \rangle|^2 = 0,$$

so that
$$\|w_N\|^2 = \langle w, w_N \rangle \leq \|w_N\|\|w\|.$$
It follows that $\sum_{i=1}^{+\infty} |\langle v_i, w \rangle|^2$ converges, and therefore that w_N is Cauchy: by completeness of V, it therefore converges to some element $\hat{w} = \sum_{i=1}^{+\infty} \langle v_i, w \rangle\, v_i$. Finally, for each fixed index $j \geq 1$ and each $N \geq j$, orthogonality gives

$$\big\langle w - w_N,\ v_j \big\rangle = \langle w, v_j \rangle - \sum_{i=1}^{N} \langle v_i, w \rangle\, \langle v_i, v_j \rangle = \langle v_j, w \rangle\; - \langle v_j, w \rangle\; = 0.$$

But now finally

$$|\langle w - \hat{w}, v_j \rangle| = |\langle w - w_N, v_j \rangle + \langle w_N - \hat{w}, v_j \rangle| \leq \|w_N - \hat{w}\|\|v_j\|$$

which goes to zero as $n \to \infty$, proving the result. □

Exercise 5.

1. *Show that in any inner product space $(V, \langle \cdot, \cdot \rangle)$ with orthonormal basis $(v_i)_{i\geq 1}$, for any $w \in V$ the norm*

$$\|w - \sum_{i=1}^{n} c_i v_i\|$$

is (strictly) minimized by $c_i = \langle v_i, w \rangle$.

2. *Using this, show the second item of Lemma 3.16, i.e. that for $(V, \langle \cdot, \cdot \rangle)$ an inner product space admitting an orthonormal basis $(v_i)_{i\geq 1}$, the writing $w = \sum_{i\geq 1} a_i v_i$ of any $w \in V$ is such that each a_i is uniquely determined, and actually equal to $\langle v, v_i \rangle$.*

Proof. 1. Using linearity of the inner product, we get

$$\begin{aligned}\langle w - \sum_{i=1}^{n} c_i v_i, w - \sum_{i=1}^{n} c_i v_i \rangle &= \langle w, w \rangle - 2\langle w, \sum_{i=1}^{n} c_i v_i \rangle + \langle \sum_{i=1}^{n} c_i v_i, \sum_{i=1}^{n} c_i v_i \rangle \\ &= \langle w, w \rangle - 2\sum_{i=1}^{n} c_i \langle w, v_i \rangle + \sum_{i=1}^{n} c_i^2 \langle v_i, v_i \rangle \\ &= \langle w, w \rangle - 2\sum_{i=1}^{n} c_i \langle w, v_i \rangle + \sum_{i=1}^{n} c_i^2\end{aligned}$$

This is a convex function in $c_1, \ldots, c_n$, hence taking partial derivatives with respect to c_i and setting the result equal to zero, we get $c_i = \langle v_i, w \rangle$ for $i \in [n]$, which minimizes the expression. Alternatively, it is a second order polynomial, for which it is clear that the choice $c_i = \langle v_i, w \rangle$ is a strict minimizer.

2. Suppose there exists another writing $w = \sum_{i\geq 1} b_i v_i$, with some $K \geq 1$ such that $b_K \neq \langle w, v_K \rangle$. By 1), we know that for some $\delta > 0$ we have

$$\left\| w - \sum_{i=1}^{K} b_i v_i \right\| = \left\| w - \sum_{i=1}^{K} \langle w, v_i \rangle v_i \right\| + \delta.$$

But now for any $N \geq K$,

$$\begin{aligned}\left\| \sum_{i=1}^{N} b_i v_i - \sum_{i=1}^{N} \langle w, v_i \rangle v_i \right\|^2 &= \left\| \sum_{i=1}^{K} b_i v_i - \sum_{i=1}^{K} \langle w, v_i \rangle v_i \right\|^2 + \left\| \sum_{i=K+1}^{N} b_i v_i - \sum_{i=K+1}^{N} \langle w, v_i \rangle v_i \right\|^2 \\ &\geq \left(\left\| w - \sum_{i=1}^{K} b_i v_i \right\| - \left\| w - \sum_{i=1}^{K} \langle w, v_i \rangle v_i \right\| \right)^2 = \delta^2\end{aligned}$$

which is a contradiction to the convergence of both series on the left-hand side to w in V. $\square$

Exercise 6. *Let $l^2(\mathbb{N})$ denote the set of all real-valued sequences $\bar{c} = (c_i)_{i\geq 1}$ with $\sum_{i\geq 1} c_i^2 < \infty$. Show that equipping it with (coordinate-wise) addition and inner product $\langle \bar{a}, \bar{b} \rangle = \sum_{i\geq 1} a_i b_i$ turns it into an inner product space.*

Proof. We check that all properties of the inner product hold. Symmetry follows from the fact that $a_i b_i = b_i a_i$ for all $i \geq 1$. For linearity, note that for $\bar{a}, \bar{b} \in l^2(\mathbb{N})$, $\sum_{i\geq 1} |a_i b_i| \leq \sum_{i\geq 1} a_i^2 + b_i^2 < \infty$ which shows that the product of the sequences converge absolutely. This enables us to split the sum in the following way

$$\begin{aligned}\langle \lambda \bar{a} + \mu \bar{b}, \bar{c} \rangle &= \sum_{i \geq 1} (\lambda a_i + \mu b_i) c_i \\ &= \lambda \sum_{i \geq 1} a_i c_i + \mu \sum_{i \geq 1} b_i c_i \\ &= \lambda \langle \bar{a}, \bar{c} \rangle + \mu \langle \bar{b}, \bar{c} \rangle.\end{aligned}$$

Finally, non-negativity follows from $\langle \bar{a}, \bar{a} \rangle = \sum_{i \geq 1} a_i^2 \geq 0$. Note also that by Cauchy-Schwartz, if $c, c' \in L^2(\mathbb{N})$, $\lambda, \mu \in \mathbb{R}$, then $\mu c + \lambda c' \in L^2(\mathbb{N})$ as $\|\mu c + \lambda c'\|^2 = \langle \mu c + \lambda c', \mu c + \lambda c' \rangle \leq \mu^2 \|c\|^2 + \lambda^2 \|c'\|^2 < +\infty$ by assumption. □