Exercise sheet 11

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1 (Translations and dilations)**.** *Let $f \in L^1(\mathbb{R}), \alpha \in \mathbb{R}, \mu \in \mathbb{R}^*$. Prove that*

$$\int_{\mathbb{R}} f(x-\alpha)\mathrm{d}\lambda(x) = \int_{\mathbb{R}} f(x)\mathrm{d}\lambda(x),$$
$$\int_{\mathbb{R}} f(\mu x)\mathrm{d}\lambda(x) = \frac{1}{\mu}\int_{\mathbb{R}} f(x)\mathrm{d}\lambda(x).$$

Proof. Consider first E a measurable set. From translation invariance and the dilation property of the Lebesgue measure (see the construction in the notes), which follows from $\lambda([a-\alpha, b-\alpha]) = (a-\alpha)-(b-\alpha) = \lambda([a,b])$ and $\lambda([\mu a, \mu b]) = \mu b - \mu a = \mu\lambda([a,b])$, one has that the two equations in the statement hold for $f = \mathbf{1}_E$. More generally, it follows by linearity of these equations for all f finite simple functions $f = \sum_{i=0}^n \alpha_i \mathbf{1}_{E_i}$:

$$\int_{\mathbb{R}} f(x-\alpha)\mathrm{d}\lambda(x) = \int_{\mathbb{R}} \sum_{i=0}^n \alpha_i \mathbf{1}_{E_i+\alpha}\mathrm{d}\lambda(x) = \sum_{i=0}^n \alpha_i\lambda(E_i+\alpha) = \sum_{i=0}^n \alpha_i\lambda(E_i) = \int_{\mathbb{R}} f(x)\mathrm{d}\lambda(x),$$

$$\int_{\mathbb{R}} f(\mu x)\mathrm{d}\lambda(x) = \int_{\mathbb{R}} \sum_{i=0}^n \alpha_i \mathbf{1}_{E_i/\mu}\mathrm{d}\lambda(x) = \sum_{i=0}^n \alpha_i\lambda(E_i/\mu) = \sum_{i=0}^n \alpha_i\frac{1}{\mu}\lambda(E_i) = \frac{1}{\mu}\int_{\mathbb{R}} f(x)\mathrm{d}\lambda(x).$$

To conclude in the general case, take $f \in L^1(\mathbb{R})$, and take an increasing sequence of simple functions $(f_n)_{n\geq 1}$ such that $f_n \xrightarrow[n\to\infty]{} f$ in $L^1(\mathbb{R})$. One has for the first property:

$$\int_{\mathbb{R}} f_n(x-\alpha)\mathrm{d}\lambda(x) = \int_{\mathbb{R}} f_n(x)\mathrm{d}\lambda(x),$$

and the right-hand side converges to $\int_{\mathbb{R}} f(x-\alpha)\mathrm{d}\lambda(x)$ as $n \to \infty$ by the monotone convergence theorem (which also shows that $f(\cdot - \alpha)$) is integrable), and the right-hand side to $\int_{\mathbb{R}} f(x)\mathrm{d}\lambda(x)$ as $n \to \infty$. The second property follows almost identically. □

Exercise 2. *The aim is to calculate* $I = \int_{(0,\infty)} \exp(-x)\frac{\sin^2(x)}{x}d\lambda(x)$. *To do this, we define* $f(x,y) = \exp(-x)\sin(2xy)$ *and use Fubini:*

- *Show that $f(x,y)$ is integrable over $(0,\infty)\times[0,1]$*
- *Show that when first integrating y over $[0,1]$ we obtain exactly I.*
- *On the other hand, calculate explicitly the integral by first integrating over x. Integration by parts might be useful.*

Proof. Recall that we have seen with Fubini's theorem that $g = f\mathbf{1}_{(0,+\infty)\times(0,1)}$ is integrable on $\mathbb{R}^2$ if the integral of its absolute value is finite when integrating first one variable, then the other. Here, we have

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x,y)\mathbf{1}_{(0,+\infty)\times(0,1)}|\mathrm{d}\lambda(y)\right)\mathrm{d}\lambda(x) \leq \int_{\mathbb{R}} \exp(-x)\mathbf{1}_{(0,+\infty)}\mathrm{d}\lambda(x) < +\infty$$

which is finite as seen in previous exercise sheets.

For the second part of the exercise we first integrate $f(x,y)$ over y for $x \in (0,+\infty)$ fixed. Since $f(x,\cdot)$ is continuous on $[0,1]$ we can use standard integration techniques.

$$\begin{aligned}\int_{[0,1]} f(x,y)d\lambda(y) &= \exp(-x)\int_{[0,1]} \sin(2xy)d\lambda(y)\\ &= \exp(-x)\Big[\frac{1}{2x}(-\cos(2xy)\Big]_{y=0}^{y=1}\\ &= \exp(-x)\frac{1-\cos(2x)}{2x} = \exp(-x)\frac{\sin^2(x)}{x}.\end{aligned}$$

Integrating the above expression over x, we recover I, which proves the second part of the exercise.

For the third part, we use integration by parts twice. For $N \geq 1$, we have that

$$\begin{aligned}\int_0^N \exp(-x)\sin(2xy)d\lambda(x) &= \big[-\exp(-x)\sin(2xy)\big]_{x=0}^{x=N} - \int_0^N -\exp(-x)2y(-\cos(2xy))d\lambda(x)\\ &= -2y\int_0^N \exp(-x)\cos(2xy)d\lambda(x) - \exp(-N)\sin(2Ny).\end{aligned}$$

We apply integration by parts again.

$$\begin{aligned}\int_0^N \exp(-x)\cos(2xy)d\lambda(x) &= \big[-\exp(-x)\cos(2xy)\big]_{x=0}^{x=N} - \int_0^N -\exp(-x)2y\sin(2xy)d\lambda(x)\\ &= -1 + 2y\int_0^N \exp(-x)\sin(2xy)d\lambda(x) - \exp(-N)\cos(2Ny).\end{aligned}$$

We can now take the limit $N \to \infty$ as all terms individually converge, and the integral does by the dominated convergence theorem: $\exp(-\cdot)\sin(2\cdot y)\mathbf{1}_{(0,N)}$ converges pointwise to $\exp(-\cdot)\sin(2\cdot y)\mathbf{1}_{(0,+\infty)}$, and is bounded in absolute value by $\exp(-\cdot)\mathbf{1}_{(0,+\infty)}$ which is integrable.

We therefore get that

$$\int_0^\infty \exp(-x)\sin(2xy)d\lambda(x) = \frac{2y}{1+4y^2}.$$

Now, using Fubini we conclude that

$$I = \int_0^1 \frac{2y}{1+4y^2}d\lambda(y) = \Big[\frac{\log\big(1+4y^2\big)}{4}\Big]_0^1 = \frac{\log(5)}{4},$$

using standard integration techniques here again since the integrand is continuous. □

Remark 1. *It's important here to carry out the integration by parts on a bounded interval first, then taking the limit using convergence theorems, to avoid any integration issue. It's the same for functions only continuous over an open interval (for instance, $f : x \mapsto x^{-\alpha}$ on $[0,1]$).*

Exercise 3 (Convolutions, I)**.** *Fix g a bounded integrable function on $\mathbb{R}$ and consider the convolution product $f \star g$ on $L^1(\mathbb{R})$ (i.e. for $f \in L^1$), defined by*

$$(f \star g)(x) := \int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y.$$

1. *Show that $f \star g$ on $L^1(\mathbb{R})$ is well-defined. Is it also well-defined on $\mathcal{L}^1(\mathbb{R})$?*

2. *Show that the convolution product is bilinear, and commutative if f, g are both bounded and in L^1.*

Proof. 1. We have to show that the integral makes sense, i.e. that $f(\cdot)g(x-\cdot)$ is integrable for all $x \in \mathbb{R}$. By assumption, let $C > 0$ be such that $|g| \leq C$. Note first that

$$\left|\int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y\right| \leq C\int_{\mathbb{R}} |f(x-y)|\mathrm{d}y = C\int_{\mathbb{R}} |f(y)|\mathrm{d}y = C\|f\|_{L^1} < +\infty.$$

In the first equality, we used the translation invariance of the Lebesgue measure. Now by a change of variables (a translation once again) this shows that $\int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y$ is also well-defined and therefore the convolution product makes sense. We now need to see whether $f*g \in L^1$. We claim that $(x,y) \mapsto f(y)g(x-y)$, which is clearly measurable as a composition and product of measurable functions, is integrable over $\mathbb{R}^2$. Indeed,

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(y)g(x-y)|\mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{R}} |f(y)|\left(\int_{\mathbb{R}} |g(x-y)|\mathrm{d}x\right)\mathrm{d}y = \|f\|_1\|g\|_1 < +\infty$$

where we have used translation invariance. It now follows from Fubini's theorem (or rather, the converse thereof) that $(x,y) \mapsto f(y)g(x-y)$ is integrable and its integral, bounded by $\|f\|_1\|g\|_1$, can also be computed as

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y\right)\mathrm{d}x = \int_{\mathbb{R}}(f*g)(x)\mathrm{d}x.$$

Consequently, $f*g$ is indeed in L^1 (and $\|f*g\|_1 \leq \|f\|_1\|g\|_1$).

Lastly, to check that the convolution product is well-defined on $\mathcal{L}^1(\mathbb{R})$, we must check that if $f, h \in L^1(\mathbb{R})$ are such that $f = h$ almost everywhere, then $f*g = h*g$. Indeed, for all $x \in \mathbb{R}$,

$$|(f*g)(x) - (h*g)(x)| \leq C\int_{\mathbb{R}} |f(y) - h(y)|\mathrm{d}\lambda(y) = 0$$

since $|f-h| = 0$ a.e.

2. If $f, h \in L^1$, $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta h \in L^1$ and linearity follows directly:

$$((\alpha f + \beta h) \star g)(x) = \int_{\mathbb{R}}(\alpha f + \beta h)(y)g(x-y)\mathrm{d}y = \alpha(f \star g)(x) + \beta(h \star g)(x).$$

Commutativity follows from a change of variables:

$$(f \star g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\mathrm{d}y = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y = (g \star f)(x).$$

□

Exercise 4. *Find a function $f : \mathbb{R} \to \mathbb{R}$ that is integrable but not square integrable. Find also a function $f : \mathbb{R} \to \mathbb{R}$ that is square-integrable, but not integrable.*

However, prove that if E satisfies $\lambda(E) < \infty$, then $L^1(E) \supset L^2(E)$.

Proof. Consider $f : x \mapsto x^{-1/2}\mathbf{1}_{0<x<1}$, $g : x \mapsto 1/x \cdot \mathbf{1}_{x>1}$. We have already seen in previous exercise sheets that f and g^2 are Lebesgue (and Riemann) integrable, whereas f^2 and g are not.

If $\lambda(E) < \infty$, then for any $f \in L^2(E)$, we can decompose:

$$|f| = |f|\mathbf{1}_{|f|\leq 1} + |f|\mathbf{1}_{|f|>1} \leq |f|\mathbf{1}_{|f|\leq 1} + |f|^2\mathbf{1}_{|f|>1} \leq 1 + |f|^2,$$

and the latter is integrable by the assumptions, with integral $\lambda(E) + \|f\|_2^2$ by linearity. We conclude that $|f|$ is integrable (i.e. $f \in L^1$) and $\|f\|_1 \leq \lambda(E) + \|f\|_2^2$. □

Exercise 5. *Show that for all $p \geq 1$, there exists c_p such that for all $f, g \in L^p$,*

$$\int |f+g|^p \mathrm{d}\lambda \leq c_p \left(\int |f|^p \mathrm{d}\lambda + \int |g|^p \mathrm{d}\lambda \right)$$

<u>Hint:</u> you might want to use the inequality that for all $p \geq 1$ there exists c_p such that for all $a, b > 0$:

$$(a+b)^p \leq c_p(a^p + b^p).$$

Prove this inequality by for example using convexity of $x \mapsto x^p$ or otherwise.

Proof. Let $p \geq 1$ and suppose $f, g \in L^p(\mathbb{R})$, so that

$$\int_{\mathbb{R}} |f(x)|^p \, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} |g(x)|^p \, dx < \infty.$$

Since $t \mapsto t^p$ is convex on $[0, \infty)$, for all $a, b \geq 0$ we have

$$(a+b)^p = \left(2 \cdot \tfrac{a+b}{2}\right)^p = 2^p \left(\tfrac{a+b}{2}\right)^p \leq 2^p \cdot \tfrac{a^p + b^p}{2} = 2^{p-1}(a^p + b^p).$$

Thus,

$$|f(x) + g(x)|^p \;\leq\; 2^{p-1}\big(|f(x)|^p + |g(x)|^p\big).$$

Integrating over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} |f+g|^p \, dx \;=\; \int_{\mathbb{R}} |f(x)+g(x)|^p \, dx \;\leq\; 2^{p-1} \int_{\mathbb{R}} \big(|f(x)|^p + |g(x)|^p\big) \, dx < \infty,$$

as wanted. □