Exercise sheet 10

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1 (Some properties, revisited)**.** *Recall that a property P is said to hold almost everywhere in $\mathbb{R}$, or for almost all $x \in \mathbb{R}$, if the set of $x \in \mathbb{R}$ for which P does not hold is measurable and of Lebesgue measure zero. Show the following form of linearity:*

Theorem. *Let f, g, h be measurable, integrable and suppose that $h = f + g$ holds almost everywhere. Then h is integrable and*

$$\int h d\lambda = \int f d\lambda + \int g d\lambda.$$

Show also the following stronger formulation of the monotone convergence theorem:

Theorem. *Let $(f_n)_{n\geq 1}$ be a sequence of measurable and integrable functions from $\mathbb{R}$ to $\mathbb{R}^+$, that is almost everywhere increasing, i.e. such that for all $n \geq 1$, almost everywhere[1] $f_n \leq f_{n+1}$. Suppose also that there exists f measurable such that $f_n \xrightarrow[n\to\infty]{} f$ almost everywhere. Then*

$$\lim_{n\to\infty} \int f_n \mathrm{d}\lambda = \int f \mathrm{d}\lambda.$$

Proof. The idea of this proof is to exploit that the Lebesgue measure does not see the difference between cleverly-chosen modifications of the given functions. More precisely, let

$$N \;=\; \{x \in \mathbb{R} : h(x) \neq f(x) + g(x)\}.$$

By assumption N is measurable and $\lambda(N) = 0$. Define modified functions

$$f'(x) = \begin{cases} f(x), & x \notin N, \\ 0, & x \in N, \end{cases} \quad g'(x) = \begin{cases} g(x), & x \notin N, \\ 0, & x \in N, \end{cases} \quad h'(x) = f'(x) + g'(x)$$

(these are modifications of f and g, respectively, not to be confused with their derivatives). Since N has measure zero, each of f', g', h' is measurable, and each is further integrable by monotonicity as f, g, h are integrable. Furthermore, property 3 of Lemma 2.26 and linearity imply that

$$\int f'\, d\lambda = \int f\, d\lambda, \quad \int g'\, d\lambda = \int g\, d\lambda, \quad \int h'\, d\lambda = \int h\, d\lambda.$$

But now $h' = f' + g'$ everywhere on $\mathbb{R}$, by linearity of the Lebesgue integral,

$$\int h'\, d\lambda = \int (f' + g')\, d\lambda = \int f'\, d\lambda + \int g'\, d\lambda.$$

Substituting back the equalities of the integrals shows

$$\int h\,d\lambda = \int f\,d\lambda + \int g\,d\lambda,$$

as desired.

We now extend the monotone convergence theorem. Since for each n the inequality $f_n \le f_{n+1}$ and the pointwise convergence $f_n \to f$ hold only almost everywhere, let us first remove a null set.

By hypothesis there exists a measurable set

$$N \subset \mathbb{R}, \qquad \lambda(N) = 0,$$

such that for every $x \notin N$ and every $n \ge 1$ we have[2]

$$f_n(x) \le f_{n+1}(x) \quad \text{and} \quad \lim_{n\to\infty} f_n(x) = f(x).$$

Define for $n \ge 1$ the "cut-off" functions

$$\tilde{f}_n(x) \;=\; f_n(x)\,\mathbf{1}_{\mathbb{R}\setminus N}(x), \qquad \tilde{f}(x) \;=\; f(x)\,\mathbf{1}_{\mathbb{R}\setminus N}(x).$$

Each $\tilde{f}_n$ (and $\tilde{f}$) is measurable, nonnegative or integrable just as f_n (resp. f) was, and by the same argument as in the previous sub-exercise

$$\int \tilde{f}_n\,d\lambda \;=\; \int f_n\,d\lambda, \qquad \int \tilde{f}\,d\lambda \;=\; \int f\,d\lambda$$

(changing a function on a set of measure zero does not affect its integral).

Moreover, by construction $\tilde{f}_n(x) \le \tilde{f}_{n+1}(x)$ for every $x \in \mathbb{R}$, and $\tilde{f}_n(x) \to \tilde{f}(x)$ for every $x \in \mathbb{R}$. Hence the "classical" Monotone Convergence Theorem applies to the sequence $(\tilde{f}_n)$:

$$\lim_{n\to\infty} \int \tilde{f}_n\,d\lambda \;=\; \int \Big(\lim_{n\to\infty} \tilde{f}_n\Big)\,d\lambda \;=\; \int \tilde{f}\,d\lambda.$$

Substituting back $\int \tilde{f}_n = \int f_n$ and $\int \tilde{f} = \int f$ yields the desired conclusion:

$$\lim_{n\to\infty} \int f_n\,d\lambda \;=\; \int f\,d\lambda.$$

□

Exercise 2 (Reminder: switching sums)**.** *Provide examples of double sequences $(a_{n,m})_{n,m\in\mathbb{N}}$ such that one of the limits below converges, but not the others; or that they all converge but to different limits:*

$$1)\quad \sum_{n=1}^{+\infty}\left(\sum_{m=1}^{+\infty} a_{n,m}\right) := \lim_{N\to\infty}\sum_{n=1}^{N}\left(\lim_{M\to\infty}\sum_{m=1}^{M} a_{n,m}\right)$$

$$2)\quad \sum_{m=1}^{+\infty}\left(\sum_{n=1}^{+\infty} a_{n,m}\right) := \lim_{M\to\infty}\sum_{m=1}^{M}\left(\lim_{N\to\infty}\sum_{n=1}^{N} a_{n,m}\right)$$

$$3)\quad \lim_{K\to\infty}\sum_{m=1}^{K}\sum_{n=1}^{K} a_{n,m}$$

On the other hand, prove that if $(a_{n,m})_{n,m\in\mathbb{N}}$ is absolutely summable, i.e. if one of these limits above exists when replacing $a_{m,n}$ with $|a_{m,n}|$, then the others do as well and all the results are the same.

[2] If one considered that for each $n \ge 1$, there is a measurable set N_n with $\lambda(N_n) = 0$ and $f_n(x) \le f_{n+1}(x)$ for all $x \notin N_n$, then the union $\bigcup_{n\ge1} N_n$ is still measurable and of Lebesgue measure zero by subadditivity.

Proof. Define

$$a_{n,m} = \begin{cases} +1, & m = 2n, \\ -1, & m = 2n-1, \\ 0, & \text{otherwise} \end{cases}$$

(make a drawing of $\mathbb{R}^2$ partitioned into a grid with the natural numbers to get a visual understanding of where this is going). Then:

1. For each fixed $n \in \mathbb{N}$,

$$\sum_{m=1}^{\infty} a_{n,m} = a_{n,2n-1} + a_{n,2n} = (-1) + (+1) = 0.$$

   Hence

$$\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} a_{n,m}\Big) = \sum_{n=1}^{\infty} 0 = 0.$$

2. For each fixed m, there is at most one n with $a_{n,m} \neq 0$. In fact, it is

$$n = \begin{cases} \frac{m+1}{2}, & m \text{ odd}, \\ \frac{m}{2}, & m \text{ even} \end{cases}$$

   and

$$\sum_{n=1}^{\infty} a_{n,m} = \begin{cases} -1, & m \text{ odd}, \\ +1, & m \text{ even}. \end{cases}$$

   Therefore, the partial sums

$$\sum_{m=1}^{M}\Big(\sum_{n=1}^{\infty} a_{n,m}\Big) = 1 - 1 + 1 - 1 + \cdots \quad (M \text{ terms})$$

   do not converge as $M \to \infty$. Hence, $\sum_{m=1}^{\infty}\big(\sum_{n=1}^{\infty} a_{n,m}\big)$ does not exist.

3. Double sum over the "square": For each $K \in \mathbb{N}$, consider

$$S_K = \sum_{n=1}^{K}\sum_{m=1}^{K} a_{n,m}.$$

   Note that

$$S_K = \sum_{n=1}^{K}\big(a_{n,2n-1} + a_{n,2n}\big),$$

   where we take $a_{n,2n-1}$ only when $2n-1 \leq K$, and similarly for $2n \leq K$. One can see that

$$S_{2L} = 0, \qquad S_{2L+1} = -1.$$

   Hence $\lim_{K\to\infty} S_K$ does not exist, as there exist two subsequences with different limits.

On the other hand, assume

$$A := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{n,m}| < \infty.$$

We first note that $A' = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left|a_{n,m}\right| < \infty$ since

$$\sum_{m=1}^{M} \sum_{n=1}^{N} |a_{n,m}| \leq \sum_{m=1}^{M} \sum_{n=1}^{+\infty} |a_{n,m}| \leq \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} |a_{n,m}|$$

and we can first take the limit $n \to \infty$, then $m \to \infty$ on the left term, which is jointly increasing. This implies that $A' \leq A$ and the symmetric argument shows that $A \leq A'$.

For the signed version, set

$$T_{N,M} = \sum_{n=1}^{N} \sum_{m=1}^{M} a_{n,m},$$

and notice that since absolute convergence of a series implies convergence, for all $N \geq 1$ fixed we have that $T_{N,M} \xrightarrow[M \to \infty]{} \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_{n,m}$. Furthermore, $(\lim_{M\to\infty} T_{N,M})_{N\geq 1} = (\left|\sum_{n=1}^{N} \sum_{m=1}^{+\infty} a_{n,m}\right|)_{N\geq 1}$ is absolutely convergent by finiteness of A, so it again implies convergence of $\lim_{N\to\infty} \lim_{M\to\infty} T_{N,M} = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} a_{n,m} =: B$. Similarly, $\lim_{M\to\infty} \lim_{N\to\infty} T_{M,N} = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} a_{n,m} =: B'$ is well-defined and it remains to be shown that $B = B'$.

Given $\varepsilon > 0$, choose N_0 so large that

$$\sum_{n>N_0} \sum_{m=1}^{\infty} \left|a_{n,m}\right| < \frac{\varepsilon}{2}, \qquad \sum_{m>N_0} \sum_{n=1}^{\infty} \left|a_{n,m}\right| < \frac{\varepsilon}{2}.$$

Now, fix any $N \geq N_0$ and let $M \geq N$. Then

$$\begin{aligned}
\left|T_{N,M} - T_{M,N}\right| &= \left|\sum_{n=1}^{N} \sum_{m=1}^{M} a_{n,m} - \sum_{m=1}^{N} \sum_{n=1}^{M} a_{n,m}\right| \\
&\leq \sum_{n=1}^{N} \sum_{m=N+1}^{M} \left|a_{n,m}\right| + \sum_{m=1}^{N} \sum_{n=N+1}^{M} \left|a_{n,m}\right| \\
&\leq \sum_{m=N_0+1}^{M} \sum_{n=1}^{N} \left|a_{n,m}\right| + \sum_{n=N_0+1}^{M} \sum_{m=1}^{N} \left|a_{n,m}\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Taking the limit in the previous equation (since it exists) yields

$$\left|\sum_{n=1}^{N} \sum_{m=1}^{+\infty} a_{n,m} - \sum_{m=1}^{N} \sum_{n=1}^{+\infty} a_{n,m}\right| \leq \varepsilon$$

for all $N \geq N_0$, which shows that $L = L'$ by arbitrariness of ε. The argument for the diagonal summation is identical.

□

Exercise 3. *Let* $f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ *be (Borel-)measurable. Then for any* $0 < m < n$ *and any* $(x_1, \ldots, x_m) \in \mathbb{R}^m$*, we have that* $f_{x_1,\ldots,x_m} : \mathbb{R}^{n-m} \to \mathbb{R}$, $f_{x_1,\ldots,x_m}(y_1, \ldots, y_{n-m}) := f(x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ *is also measurable.*

Proof. The function $g : \mathbb{R}^{n-m} \to \mathbb{R}^n, (y_1, \ldots, y_{n-m}) \mapsto (x_1, \ldots, x_m, y_1, \ldots y_{n-m})$ is measurable since it is continuous. Note that $f_{x_1,\ldots,x_m} = f \circ g$, so that for any Borel set $B \subset \mathbb{R}$, it holds that $f^{-1}_{x_1,\ldots,x_m}(B) = g^{-1}(f^{-1}(B))$. Since f is Borel-measurable, $f^{-1}(B)$ is a Borel set, and therefore so is $g^{-1}(f^{-1}(B))$ since g is continuous. It follows that $f_{x_1,\ldots,x_m}$ is measurable. □

Exercise 4. *Show that $f : (0,1) \to \mathbb{R}$, $x \mapsto x^\alpha$ is integrable if and only if $\alpha > -1$. What about $f : (1,+\infty) \to \mathbb{R}$, $x \mapsto x^\alpha$? Revisit the Example 2.41 in the notes of finding a function $f : (-1,1)^2 \to \mathbb{R}$ that is integrable, but so that there is some point $x \in (-1,1)$ for which $f(x,\cdot) : (-1,1) \to \mathbb{R}$ is not integrable.*

Proof. For any $\varepsilon \in (0,1)$, the function $x \mapsto x^\alpha$ on $[\epsilon, 1]$ is continuous and therefore Riemann integrable. Thus, by Prop. 2.39, it is also Lebesgue integrable and both integrals coincide. It therefore suffices to calculate the Riemann integral with the usual techniques. For any $0 < \varepsilon < 1$ we have

$$\int_\varepsilon^1 x^\alpha\, dx = \begin{cases} \dfrac{1-\varepsilon^{\alpha+1}}{\alpha+1}, & \alpha \neq -1, \\ -\ln\varepsilon, & \alpha = -1. \end{cases}$$

Let now $f : x \mapsto x^\alpha$ on $[0,1]$. If $\alpha > -1$, then by the monotone convergence theorem applied to the sequence $(f\mathbf{1}_{[1/n,1]})_{n\geq 1}$, it holds that f is integrable and

$$\int_0^1 f \mathrm{d}\lambda = \lim_{n\to\infty} \int_0^1 f_n \mathrm{d}\lambda = \frac{1}{\alpha+1} < \infty.$$

If $\alpha \leq -1$, then f cannot be integrable, since otherwise its integral would bound that of f_n for any $n \geq 1$ by monotonicity; but $(\int_0^1 f_n \mathrm{d}\lambda)$ diverges to $+\infty$. Hence,

$$f \quad \text{is integrable} \qquad \Longleftrightarrow \qquad \alpha > -1.$$

Now, consider $g : x \mapsto x^\alpha$ on $[1,\infty)$. For any $1 < R < \infty$, we can compute the Riemann integral

$$\int_1^R x^\alpha\, dx = \begin{cases} \dfrac{R^{\alpha+1}-1}{\alpha+1}, & \alpha \neq -1, \\ \ln R, & \alpha = -1. \end{cases}$$

A similar argument with the monotone convergence theorem gives that

$$g \quad \text{is integrable} \qquad \Longleftrightarrow \qquad \alpha < -1.$$

Finally, recall Exa. 2.41: Define

$$f(x,y) = \mathbf{1}_{\mathbb{Q}\cap(-1,1)}(x)\, \frac{1}{y}\, \mathbf{1}_{(-1,1)}(y), \qquad (x,y) \in (-1,1)^2.$$

Since $\mathbb{Q} \cap (-1,1)$ is countable, $(\mathbb{Q}\cap(-1,1)) \times (-1,1)$ is a countable union of segments (having Lebesgue measure 0 in $\mathbb{R}^2$), and therefore

$$\lambda\big(\{(x,y) : f(x,y) \neq 0\}\big) = \lambda\big((\mathbb{Q}\cap(-1,1)) \times (-1,1)\big) = 0.$$

Thus, $f = 0$ almost everywhere, so it is Lebesgue-integrable and

$$\int_{(-1,1)^2} f(x,y)\, dx\, dy = 0.$$

On the other hand, take $x \in \mathbb{Q} \cap (-1,1)$. Then

$$f_x(y) \;:=\; f(x,y) = \frac{1}{y}\, \mathbf{1}_{(-1,1)}(y)$$

which is not integrable as we've just seen. ∎