EPFL

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Teacher : Maria Colombo
Analysis 4 (MATH-205)
21/06/2023
Duration : 180 minutes

Student 1

SCIPER : **999000**

Do not turn the page before the start of the exam. This document is double-sided, has 20 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A4 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  $+1$ points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Correction

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Let $f \in C(\mathbb{R}/\mathbb{Z})$ be a 1-periodic function and let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ be its real Fourier coefficents, namely $a_n = 2\int_0^1 f(x)\cos(2\pi nx)dx$, $b_n = 2\int_0^1 f(x)\sin(2\pi nx)dx$.
Assume that $f(x+1/2) = -f(x)$ for all x. Which of the following is **true**?

- [ ] $a_{2m+1} = b_{2m+1} = 0$ for all $m = 1, 2, \dots$.
- [ ] $a_n = b_n = 0$ for all $n = 1, 2, \dots$.
- [x] $a_0 = 0$ and $a_{2m} = b_{2m} = 0$ for all $m = 1, 2, \dots$.
- [ ] None of the others is true.

Question 2 Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $\hat{f}(\xi) = |\xi|^{5/2}\mathrm{e}^{-|\xi|^2/4}, \hat{g}(\xi) = \mathrm{e}^{-|\xi|^2/4}$. Which of the following is **true**?

- [ ] $g \notin \mathcal{S}(\mathbb{R})$.
- [ ] $f * g \notin L^{\infty}(\mathbb{R})$
- [x] $f \notin \mathcal{S}(\mathbb{R})$.
- [ ] f and g have compact support.

Question 3 Let us consider the following PDE

$$\partial_x u - u\partial_y u = 0\,, \tag{1}$$

for $u : \mathbb{R}^2 \to \mathbb{R}$. Assume that v, w solve (1). Which of the following statement is **true**?

- [ ] For any v, w solutions of (1), $v - w$ solves (1).
- [ ] For any v, w solutions of (1), $v + w$ solves (1).
- [x] For any v solution of (1) and any $\lambda > 0$, $v_\lambda(x, y) = \lambda v(\lambda x, y)$ solves (1).
- [ ] $v(x, y) = x + xy + y^3 + x^2 y$ is a solution of (1).

Question 4 Let $f(x) = x^2 e^{-x^2}$, and denote with $\hat{f}(\xi)$ its Fourier Transform. Which of the following statements is **true**?

- [ ] $\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 1$
- [x] $\hat{f}(\xi) \in L^1 \cap L^{\infty}$
- [ ] $\hat{f}(\xi) \in L^2(\mathbb{R})$ but $\hat{f}(\xi) \notin L^1(\mathbb{R})$
- [ ] $\hat{f}(\xi)$ has compact support in $\mathbb{R}$

Question 5 For every $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) = \dfrac{1}{x^2}\mathbb{1}_{[2^{-n}, 2^{-n+1}]}$.
Which of the following statements is **true**?

- [x] $\limsup_n \int_{\mathbb{R}} f_n \, dx > \int_{\mathbb{R}} \limsup_n f_n \, dx$.
- [ ] $\lim_n \int_{\mathbb{R}} |f_n - \lim_n f_n| \, dx = 0$.
- [ ] $\lim_n \int_{\mathbb{R}} f_n \, dx < \int_{\mathbb{R}} \liminf_n f_n \, dx$.
- [ ] $\lim_n \int_{\mathbb{R}} f_n \, dx = \int_{\mathbb{R}} \lim_n f_n \, dx$.

Question 6 Consider $f_n(x) = x^{-1/3}\mathbb{1}_{[0,1/n]}$. Then $f_n \to 0$

- □ in L^1 but not in L^2.
- □ in L^4 but not in L^∞.
- ■ in L^2 but not in L^4.
- □ in L^∞.

Question 7 Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \geq 0$. Which of the following statements is **true**?

- ■ If $f \in L^3(\mathbb{R})$ then $f^{3/2} \in L^2(\mathbb{R})$.
- □ If $f \in L^2(\mathbb{R})$ then $f \in L^1(\mathbb{R})$.
- □ If $f, g \in L^1(\mathbb{R})$ then $fg \in L^1(\mathbb{R})$.
- □ If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$.

Question 8 Let $f, g \in \mathcal{S}(\mathbb{R})$ and $\hat{f}, \hat{g}$ be their Fourier transform. Which of the following inequality is **true**?

- □ $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^\infty}\|\hat{g}\|_{L^1}$
- □ $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^2}\|\hat{g}\|_{L^2}$
- ■ $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^\infty}\|\hat{g}\|_{L^2}$
- □ $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^\infty}\|\hat{g}\|_{L^\infty}$

Question 9 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : (0,1) \to \mathbb{R}$. Which of the following is **true**?

- □ If $f_n(x) \leq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then
$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$
- □ If $f_n(x) \geq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then
$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$
- □ Let $M \in \mathbb{N}$, if $f_n(x) \geq 0$ for any $x \in (0,1)$ and for any $n \leq M$, then
$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$
- ■ Let $c \in \mathbb{R}$, if $f_n(x) \leq c$ for any $x \in (0,1)$ and for any $n \in \mathbb{N}$ then
$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$

Question 10 Which of the following sets has infinite outer measure?

- ■ $\bigcup_{n=1}^{\infty} \left(n-3, \frac{(n-1)^3}{n^2}\right)$
- □ $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$
- □ $\bigcup_{n=1}^{\infty} \left(-\cos^2(\frac{1}{n}), \sin^2(\frac{1}{n})\right)$
- □ $\bigcup_{n=1}^{\infty} \left(2^n, \frac{2^{2n}+1}{2^n}\right)$

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 11: *This question is worth 5 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ■ 5

Let $A = (1, \infty) \times [0, \pi/2]$, $n \in \mathbb{N}$ and $f_n : A \to \mathbb{R}$ defined as

$$f_n(x, y) = \frac{x + \cos(2y)}{(x^2 + 1 + \frac{y}{n})^2}.$$

Compute $\lim_{n\to\infty} f_n(x, y)$ and show that

$$\lim_{n\to\infty} \int_A f_n(x, y) dx\, dy = \frac{\pi}{8}.$$

Proof: Since $y/n \to 0$ as $n \to \infty$, it is straightforward to check that $f_n(x, y) \to \frac{x+\cos(2y)}{(x^2+1)^2}$. Furthermore, $|f_n(x, y)| \le \frac{x+1}{(x^2+1)^2} \le \frac{1}{x^2+1} \in L^1(A)$ for any $n \in \mathbb{N}$. Therefore we can apply dominated convergence theorem and Fubini theorem and compute

$$\begin{aligned}\lim_{n\to\infty} \int_A f_n(x, y) dx\, dy &= \int_1^\infty \int_0^{\pi/2} \frac{x + \cos(2y)}{(x^2 + 1)^2} dy\, dx \\ &= \frac{\pi}{2} \int_1^\infty \frac{x}{(x^2 + 1)^2} dx = \frac{\pi}{4} \int_2^\infty \frac{1}{y^2} dy = \frac{\pi}{8}\end{aligned}$$

where we used that $\cos(2y)$ has average 0 in $(0, \pi/2)$.
Common error - you don't need to have $f_n \ge 0$ to apply dominated convergence theorem

Question 12: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ■5

Let $f \in L^1(\mathbb{R})$. Then, prove the following:

(a) *(2 points).*
$$\lim_{n\to\infty} \int_n^\infty |f(x)|dx = 0\,.$$

(b) *(1 point).*
$$\liminf_{x\to\infty} |f(x)| = 0\,.$$

(c) *(2 points).* There exists a function $f \in L^1(\mathbb{R})$ such that
$$\limsup_{x\to\infty} |f(x)| = \infty\,.$$

Proof:

(a) We apply the dominated convergence theorem to the sequence $f_n(x) = \mathbb{1}_{[n,\infty)}(x)|f(x)|$ which satisfies $|f_n(x)| \le |f(x)| \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$ and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$. Therefore we can compute
$$\lim_{n\to\infty} \int_n^\infty |f(x)|dx = \lim_{n\to\infty} \int_{-\infty}^\infty f_n(x)dx = \int_{-\infty}^\infty \lim_{n\to\infty} f_n(x)dx = 0\,.$$

(b) Suppose by contradiction that $\liminf_{x\to\infty} |f(x)| = c > 0$, then by definition of the liminf there exists $x_0 > 0$ such that $|f(x)| \ge c/2$ for any $x \ge x_0$. Therefore,
$$\int_{\mathbb{R}} |f(x)|dx \ge \int_{x_0}^\infty |f(x)|dx \ge \int_{x_0}^\infty c/2dx = \infty\,,$$
which is a contradiction with $f \in L^1$.

(c) Consider $f(x) = \sum_{n=1}^\infty n\mathbb{1}_{[n,n+2^n]}(x)(x)$. Applying the monotone convergence theorem to $f_N(x) = \sum_{n=1}^N n\mathbb{1}_{[n,n+2^{-n}]}(x)$ it is straightforward to verify that
$$\|f\|_{L^1} \le \sum_{n=1}^\infty \int_{\mathbb{R}} n\mathbb{1}_{[n,n+2^n]}(x)dx = \sum_{n=1}^\infty n2^{-n} \le \sum_{n=1}^\infty 2^{-n/2} < \infty\,.$$

Common error

- being in L^1 does not mean your limit at infinity is 0

Question 13: *This question is worth 5 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ■ 5

Consider two non-negative simple functions $f, g : [0,1] \to \mathbb{R}$, namely $f(x) = \sum_{i=1}^N a_i \mathbb{1}_{A_i}(x)$, $g(x) = \sum_{i=1}^N c_i \mathbb{1}_{C_i}(x)$ where $N \in \mathbb{N}$ and $a_i \geq 0\,, c_i \geq 0$, $C_i, A_i \subset [0,1]$ are measurable sets for any $i = 1, \dots, N$. We associate to f, g the two functions

$$F(x) = \sum_{i=1}^N a_i \mathbb{1}_{[0,m(A_i)]}(x), \qquad G(x) = \sum_{i=1}^N c_i \mathbb{1}_{[0,m(C_i)]}(x),$$

Prove the following:

(a) *(3 points).*

$$\int_0^1 |f(x)|dx = \int_0^1 |F(x)|dx$$
$$\int_0^1 f(x)g(x)dx \leq \int_0^1 F(x)G(x)dx$$

(b) *(2 points).* Finally, for $N = M = 1$ and $A_1 \cap C_1 = \emptyset$ prove that

$$\int_0^1 |F(x) - G(x)|dx \leq \int_0^1 |f(x) - g(x)|dx.$$

Proof:

(a) For the first identity we simply observe that it follows by definition since $m(E) = m([0, m(E)])$ for any $E \subset [0,1]$.

From now on we use the notation $I_E = [0, m(E)]$ for any $E \subset [0,1]$. To prove the second identity, we observe that

$$\int_{\mathbb{R}} f(x)g(x)dx = \sum_{i=1}^N \sum_{j=1}^M c_i a_j m(A_j \cap C_i).$$

But we also know that

$$m(A_j \cap C_i) \leq \min\{m(A_j), m(C_i)\} = m(I_{A_j} \cap I_{C_i}).$$

Therefore, since $c_i, a_j \geq 0$, we have

$$\int_{\mathbb{R}} f(x)g(x)dx \leq \sum_{i=1}^N \sum_{j=1}^M c_i a_j m(I_{A_j} \cap I_{C_i}) = \int_{\mathbb{R}} F(x)G(x)dx.$$

(b) We observe that the following equality holds

$$\int_0^1 |f(x) - g(x)| = am(A \setminus C) + cm(C \setminus A) + |a - c| m(A \cap C) = am(A) + cm(C)\,,$$

where in the last we used $A \cap C = \emptyset$. We suppose, without loss of generality, that $m(A) \geq m(C)$. Then,

$$\int_0^1 |F(x) - G(x)|dx = \int_{[0,m(C)]} |F(x) - G(x)|dx + \int_{[m(C),m(A)]} |F(x) - G(x)|dx$$
$$= |a - c| m(C) + a(m(A) - m(C))\,.$$

We finally notice that

$$|a - c| m(C) + a(m(A) - m(C)) = am(A) + (|a - c| - a)m(C) \leq am(A) + cm(C)$$

where the last holds thanks to the triangular inequality.

Common errors

- The value of the integral depending on x.
- Double summation with wrong indices.

CORRECTION

Question 14: *This question is worth 8 points.*

□ 0 □ 1 □ 2 □ 3 □ 4 □ 5 □ 6 □ 7 ■ 8

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $\sum_{n=-\infty}^{\infty}|a_n|<\infty$ and define $\varphi(x)=\sum_{n=-\infty}^{\infty}a_n e^{inx}$. Consider the PDE

$$\begin{cases} \partial_t u = 2\partial_{xx}u + u - \varphi \qquad (t,x)\in(0,\infty)\times(-\pi,\pi)\,, \\ u(0,x)\equiv 0 \qquad \text{for all } x\in[-\pi,\pi) \\ u(t,-\pi)=u(t,\pi) \qquad \text{for all } t>0\,, \end{cases} \tag{2}$$

(a) *(1 point).* Prove that the sequence $\{\varphi_N\}_{N\in\mathbb{N}}$ defined as $\varphi_N=\sum_{n=-N}^{N}a_n e^{inx}$ is a Cauchy sequence in $C^0[-\pi,\pi]$ and deduce that $\varphi\in C^0[-\pi,\pi]$.

(b) *(3 points).* Write $u(t,x)=\sum_{n=-\infty}^{\infty}b_n(t)e^{inx}$ and find from (2) an ODE satisfied formally for $b_n(t)$ in terms of a_n. Finally, compute explicitly $b_n(t)$.

(c) *(4 points).* Prove that the solution u found in the previous point is such that $u,\partial_t u,\partial_{xx}u\in C^0((0,\infty)\times[-\pi,\pi))$ and u solves (2) and that $u(t,x)\to 0$ as $t\to 0^+$ uniformly in x.

Hint: Recall that the solution of $\dot y = cy+d$ with $c,d\in\mathbb{R}$ and $y(0)=0$ is $y(t)=\frac{d}{c}[e^{ct}-1]$.

Proof:

(a) Using the assumption $\sum_{n=-\infty}^{\infty}|a_n|<\infty$ it is straightforward to check that φ_N is a Cauchy sequence in C^0. For any $\varepsilon>0$ there exists $\overline{N}$ such that

$$\|\varphi_N-\varphi_M\|_{C^0}\le\sum_{|n|\ge\overline{N}}|a_n|<\varepsilon$$

for any $N,M\ge\overline{N}$.

(b) Using (2) we can deduce formally that

$$\begin{cases} \dot b_n=(1-2n^2)b_n-a_n\,, \\ b_n(0)=0\,, \end{cases}$$

for any $n\in\mathbb{N}$. Therefore, a candidate solution of (2) is

$$u(t,x)=\sum_{n=-\infty}^{\infty}\frac{a_n}{1-2n^2}(1-e^{(1-2n^2)t})e^{inx}$$

(c) Define

$$u_n(x,t)=\frac{a_n}{1-2n^2}(1-e^{(1-2n^2)t})e^{inx}\,.$$

Observe that there exists $C>0$ such that

$$\|\partial_t u_n\|_{C^0}\le\frac{a_n}{1-2n^2}2n^2\le Ca_n$$

and for any $k\in\mathbb{N}$ we have

$$\partial_x^k u_n(t,x)=\frac{a_n}{1-2n^2}(in)^k(1-e^{(1-2n^2)t})e^{inx}$$

which implies that for any $k=1,2$ there exists $C>0$

$$\|\partial_x^k u_n\|_{C^0}=\left|\frac{a_n}{1-2n^2}(in)^k\right|\le Ca_n$$

from which we deduce that the series $\{(u)_N\}_N\{(\partial_t u)_N\}_N$, $\{(\partial_x u)_N\}_N$, $\{(\partial_{xx} u)_N\}_N$ defined as

$$u_N = \sum_{n=-N}^{N} u_n\,, \qquad (\partial_t u)_N = \sum_{n=-N}^{N} \partial_t u_n\,,$$
$$(\partial_x u)_N = \sum_{n=-N}^{N} \partial_x u_n\,, \qquad (\partial_{xx} u)_N = \sum_{n=-N}^{N} \partial_{xx} u_n$$

are absolutely summable and therefore the series $(u)_N$ $(\partial_t u)_N, (\partial_x u)_N, (\partial_{xx} u)_N$ are Cauchy series in C^0. We therefore we notice that

$$\sum_{n=-N}^{N} u_n \to u\,, \qquad \sum_{n=-N}^{N} \partial_t u_n \to w\,,$$
$$\sum_{n=-N}^{N} \partial_x u_n \to g\,, \qquad \sum_{n=-N}^{N} \partial_{xx} u_n \to h$$

with respect to the C^0 norm. Furthermore, as done in class we notice that $w = \partial_t u$, $g = \partial_x u$ and $h = \partial_{xx} u$. Therefore, we can justify that u is a solution of

$$\partial_t u = 2\partial_{xx} u + u - \varphi\,.$$

Finally, for the initial datum, we observe $|u_n| \leq C a_n$ and $\sum_n a_n < \infty$, therefore, for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $\sum_{n>N(\varepsilon)} a_n < \varepsilon/2$. Furthermore, $b_n(t)$ is continuous and $b_n(0) = 0$ for any n, therefore there exists $\bar{t}$ such that $|b_n(t)| < \varepsilon/2N(\varepsilon)$ for any $t \leq \bar{t}$ and therefore

$$\sum_{n=0}^{N(\varepsilon)} |b_n(t)| < \varepsilon/2\,,$$

from which we conclude

$$|u(t,x)| < \varepsilon$$

for any $t \leq \bar{t}$.

Common errors

- an infinite sum of continuous functions is not always continuous.
- proving a sum converges does not imply it converges uniformly.

CORRECTION

Question 15: *This question is worth 6 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ■ 6

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function such that

$$f(x) = \max\{0, \cos(x)\}$$

Denote with $\{a_n\}$ and $\{b_n\}$ the real Fourier coefficients of f, i.e.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

(a) (*5 points*). Compute the Fourier series of f

(b) (*1 point*). Compute the value of the series

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2-1},$$

Hint: $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$.

Proof:

(a) The Fourier series of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\,,$$

because the function is even. We thus compute the Fourier coefficients:

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x)\cos(nx)dx.$$

For $n = 0$ we have

$$a_0 = \int_{-\pi/2}^{\pi/2} \cos(x) = \frac{2}{\pi}$$

For $n \geq 1$, recall that

$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B)).$$

Therefore

$$a_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos((n+1)x) + \cos((n-1)x)dx$$

For $n = 1$ we have

$$a_1 = \frac{1}{2}$$

For $n \geq 2$ we obtain

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos((n+1)x) + \cos((n-1)x)dx \\ &= \frac{1}{\pi(n+1)}\sin((n+1)\frac{\pi}{2}) + \frac{1}{\pi(n-1)}\sin((n-1)\frac{\pi}{2}) \end{aligned}$$

Since

$$\sin((n+1)\frac{\pi}{2}) = -\sin((n-1)\frac{\pi}{2}) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} & \text{if } n \text{ is even} \end{cases}$$

Thus, when n is even we have that

$$a_n = \frac{(-1)^{n/2}}{\pi(n+1)} - \frac{(-1)^{n/2}}{\pi(n-1)} = -\frac{2}{\pi}\frac{(-1)^{n/2}}{n^2-1}.$$

Therefore

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\cos(x) - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2-1}\cos(2mx)$$

(b) In the first series we set $x = 0$ and deduce that

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2 - 1},$$

where the last equality holds because the function f is Lipschitz and therefore the series is pointwise converging. Hence

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2 - 1} = \frac{1}{2} - \frac{\pi}{4}$$

Common error - Missing a_1.

Question 16: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ■6

(a) *(2 points).* Let $g \in C_c^\infty(\mathbb{R})$. Give a formula of $\widehat{g^{(k)}}(\xi)$ in terms of $\hat{g}(\xi)$ and prove it for $k = 1$, where $g^{(k)}$ denote the k-th derivative.

(b) *(4 points).* Is there a non-zero function $f \in \mathcal{S}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \xi^k f(\xi)\, d\xi = 0 \qquad \text{for every } k \in \mathbb{N}\,?$$

Hint: for (b) consider $g \in \mathcal{S}(\mathbb{R})$ such that $f = \widehat{g}$.

Proof:

(a) State and prove it as done in class.

(b) Yes. Observe that if $f = \widehat{g}$ for some $g \in \mathcal{S}(\mathbb{R})$ then by the Fourier inversion formula it holds that

$$\int_{\mathbb{R}} x^k f(x)\, dx = \frac{1}{(2i\pi)^k} \int_{\mathbb{R}} (2i\pi\xi)^k \widehat{g}(\xi)\, d\xi = \frac{1}{(2i\pi)^k} \int_{\mathbb{R}} \mathcal{F}\left(g^{(k)}\right)(\xi) d\xi = \frac{1}{(2i\pi)^k} g^{(k)}(0)\,. \tag{3}$$

In particular, if $g \in C_c^\infty(\mathbb{R})$ is such that $g \neq 0$ and $0 \notin \operatorname{supp}(g)$, then setting $f = \widehat{g}$, we have that

- $f \neq 0$ as by Plancherel $\|f\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})} > 0\,,$
- $f \in \mathcal{S}(\mathbb{R})$ since $C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})\,,$
- $\int_{\mathbb{R}} x^k f(x) = 0$ for all $k \in \mathbb{N}_0$ by (3) and the choice of $g\,.$

Common error

- $\int \hat{g^k}(\xi) d\xi = 0 \Rightarrow g \equiv 0$ (not true for some Gaussians for instance).