First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Consider the triangle $T \subset \mathbb{R}^2$ with vertices in $A = (0,0), B = (1,1), C = (1,-1)$. What is the value of $\int_T (x+y)^2$?

- ☐ 8/9
- ■ 2/3
- ☐ It is not defined because $(x+y)^2$ is not absolutely integrable.
- ☐ 1/3

Question 2 Let $f(x) = \frac{e^{-x}}{\sqrt{x}}$. f belongs to $L^p(0,1)$ if and only if

- ■ $p \in [1,2)$
- ☐ $p \in [2,+\infty)$
- ☐ $p = 2$
- ☐ $p = +\infty$

Question 3 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Which of the following statements is **true**?

- ■ If $\liminf_{n\to\infty} \|f_n\|_{L^1} = 0$, then, for all $\varepsilon > 0$, we have $m(\{x \in \Omega : \liminf_{n\to\infty} f_n(x) < \varepsilon\}) = m(\Omega)$.
- ☐ If $\|f_n\|_{L^2} \to +\infty$, then $\|f_n\|_{L^1} \to +\infty$.
- ☐ If $\|f_n\|_{L^2} = C > 0$, then, for all $M > 0$, we have $m(\{x \in \mathbb{R}^d : |f_n(x)| \le M\}) \le \frac{C^2}{M^2}$.
- ☐ If $f_n \to 0$ almost everywhere, then $\|f_n\|_{L^1} \to 0$.

Question 4 Let $\alpha \in \mathbb{R}$. With Hölder's inequality, determine: the function $\frac{f(x)}{x^\alpha}$ belongs to $L^1(1,+\infty)$ for all $f \in L^2(1,+\infty)$ if and only if

- ☐ $\alpha \le 0$
- ☐ $0 \le \alpha \le 1/2$
- ☐ $\alpha = 1/2$
- ■ $\alpha > 1/2$

Question 5 Which of the following statements is **true**?

- ■ Every function $F : \mathbb{R} \to \mathbb{R}$ that vanishes outside the Cantor set is measurable.
- ☐ Every function $F : \mathbb{R} \to \mathbb{R}$ that vanishes on the Cantor set is measurable.
- ☐ If the function $F : \mathbb{R} \to \mathbb{R}$ is measurable, then the set $\{x \in \mathbb{R} : F(x) = x\}$ can be not measurable.
- ☐ Every function $F : \mathbb{R} \to \{0\} \cup \{1\}$ is measurable.

Midterm Exercises Solutions

Analysis IV

April 17, 2024

1 Exercises

1.1 E1 (Fat Cantor Set)

Solution:

1. The set $C_{n,\gamma}$ is Lebesgue measurable, since it consists of 2^n disjoint intervals of equal length, and has measure $m\left(C_{n,\gamma}\right) = m\left(C_{n-1,\gamma}\right) - 2^{n-1}\gamma^n$ for all $n \geq 1$, with $m\left(C_{0,\gamma}\right) = 1$.
   Hence,
   $$m\left(C_{n,\gamma}\right) = 1 - \sum_{k=1}^{n} 2^{k-1}\gamma^k = 1 - \frac{1}{2}\sum_{k=1}^{n}(2\gamma)^k = 1 - \frac{1}{2}\left(\frac{1-(2\gamma)^{n+1}}{1-2\gamma} - 1\right) = 1 - \frac{\gamma(1-(2\gamma)^n)}{1-2\gamma}\,.$$
   As a result $C_\gamma = \bigcap_{n=1}^{\infty} C_{n,\gamma}$ is Lebesgue measurable with measure
   $$m\left(C_\gamma\right) = \lim_{n\to\infty} m\left(C_{n,\gamma}\right) = 1 - \frac{\gamma}{1-2\gamma}\,.$$

2. The statement is false. Let us disprove it. For any $x \in \mathbb{R}$ we have $m(C+x) = m(C) = 0$. We then deduce that, for any sequence $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$,
   $$m\left(\bigcup_{i=1}^{\infty}(C+x_i)\right) = 0.$$
   If, by contradiction, $[0,1] \subset \bigcup_{i=1}^{\infty}(C+x_i)$, then
   $$1 = m([0,1]) \leq m\left(\bigcup_{i=1}^{\infty}(C+x_i)\right) = 0,$$
   which is absurd.

1.2 E2 (Dominated Convergence)

Solution:

1. See Lecture Notes.

2. We have that $\lim_{n\to+\infty} f_n(x)\,\mathrm{d}x = 0$. We have, for all $n \in \mathbb{N}$,
   $$\frac{e^{-x}}{n^3} \leq \frac{1}{n^2}, \quad \text{for all} \quad x \in \mathbb{R},$$
   and hence $f_n(x) \geq 0$. Then,
   $$|f_n(x)| = \left(\frac{1}{n^2} - \frac{e^{-x}}{n^3}\right)\chi_{[1,n)(x)} \leq \frac{1}{n^2}\chi_{[1,n](x)} \leq \frac{1}{x^2}\chi_{[1,n](x)} \leq \frac{1}{x^2}.$$

3. By point (2), $|f_n(x)| \leq x^{-2} \in L^1([1,+\infty))$. Hence, we can apply the dominated convergence theorem to conclude that
   $$\lim_{n\to+\infty}\int_1^{+\infty} f_n(x)\,\mathrm{d}x = \int_1^{+\infty}\lim_{n\to+\infty} f_n(x)\,\mathrm{d}x = 0.$$

4. Since $f_n \geq 0$ in $[1,+\infty)$ for every n, the sequence of partial sums is non-decreasing. Therefore, by the monotone convergence theorem, we have

$$\int_1^{+\infty} \sum_{n=1}^{+\infty} \frac{f_n(x)}{n^\alpha}\,\mathrm{d}x = \sum_{n=1}^{+\infty} \int_1^{+\infty} \frac{f_n(x)}{n^\alpha}\,\mathrm{d}x.$$

Now, we compute

$$0 \leq \int_1^{+\infty} \frac{f_n(x)}{n^\alpha}\,\mathrm{d}x \leq \frac{1}{n^\alpha}\int_1^n \left(\frac{1}{n^2} - \frac{e^{-x}}{n^3}\right)\mathrm{d}x \leq \int_1^n \frac{1}{n^{2+\alpha}}\,\mathrm{d}x \leq \frac{1}{n^{1+\alpha}}\,;$$

hence, by the comparison principle for numerical series,

$$\sum_{n=1}^{+\infty} \frac{f_n(x)}{n^\alpha} \in L^1([1,+\infty)), \qquad \text{for all} \quad \alpha > 0.$$

On the other hand, for every n sufficiently large,

$$\int_1^{+\infty} \frac{f_n(x)}{n^\alpha}\,\mathrm{d}x \geq \frac{1}{n^\alpha}\int_1^n \left(\frac{1}{n^2} - \frac{e^{-1}}{n^3}\right)\mathrm{d}x \geq \int_1^n \frac{e\,n-1}{e\,n^{2+\alpha}}\,\mathrm{d}x \geq \frac{1}{2\,e\,n^{1+\alpha}}\,;$$

hence, again by the comparison principle for numerical series,

$$\sum_{n=1}^{+\infty} \frac{f_n(x)}{n^\alpha} \notin L^1([1,+\infty)), \quad \text{for all} \quad \alpha \leq 0.$$

5. It would not have been possible to use the monotone convergence theorem. Indeed, $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative functions, but it is not monotone with respect to n a.e. in $[1,+\infty)$. Note that each f_n is non-negative and strictly positive on a set of positive measure, while $f_n \to 0$ a.e. Moreover, if $x \in [1,n]$, then

$$\begin{aligned} f_{n+1}(x) - f_n(x) &= \frac{1}{(n+1)^2} - \frac{1}{n^2} + e^{-x}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) \\ &= -\frac{2n+1}{n^2(n+1)^2} + e^{-x}\frac{3n^2+3n+1}{n^3(n+1)^3} \\ &\leq -\frac{1}{(n+1)^3} + \frac{7}{n^4} < 0, \qquad \text{for } n \text{ large enough.} \end{aligned}$$

1.3 E3 (Lebesgue Measure)

Solution: Consider a Lebesgue measurable set $E \subseteq [0,1]$ with $m(E) \geqslant \theta$.

For every $n \in \mathbb{N}$, let $E_n := \left\{x \in [0,1] : f(x) > \frac{1}{n}\right\}$. We note that $E_n \subset E_{n+1}$ for every $n \in \mathbb{N}$, and then $\lim_{n\to+\infty} m(E_n) = m\left(\bigcup_{n\geqslant 1} E_n\right) = m([0,1]) = 1$.

Then, there exists $N \in \mathbb{N}$ such that

$$0 < m(E_N^{\mathrm{c}}) < \frac{\theta}{2},$$

and, thus,

$$m(E \cap E_N) = m(E) - m(E^{\mathrm{c}} \cap E) \geq m(E) - \frac{\theta}{2} \geq \frac{\theta}{2}.$$

Now, we conclude

$$\int_E f(x)\,\mathrm{d}x \geqslant \int_{E\cap E_N} f(x)\,\mathrm{d}x \geqslant \frac{\theta}{2\,N} > 0.$$