Teacher : Maria Colombo
Analysis 4 (MATH-205)
26/04/2023
Duration : 90 minutes

SCIPER :

Do not turn the page before the start of the exam. This document is double-sided, has 8 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A5 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  $+1$ points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Pour votre examen, imprimez de préférence les documents compilés à l'aide de auto-multiple-choice.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Let $I_k \subset \mathbb{R}$ be open intervals for any $k \in \mathbb{N}$ and be such that $\mathbb{Q} \cap [0,1] \subset \bigcup_{k=0}^{\infty} I_k$. Which of the following statements is **true**?

- ■ If there exists N such that $I_k = \emptyset$ for any $k \geq N$, then $\sum_{k=0}^{\infty} m^*(I_k) \geq 1$
- ☐ $\sum_{k=0}^{\infty} m^*(I_k) > 1$
- ☐ $\sum_{k=0}^{\infty} m^*(I_k) \leq 1$
- ☐ If there exists N such that $I_k = \emptyset$ for any $k \geq N$, then $\sum_{k=0}^{\infty} m^*(I_k) < 1$

Question 2 Let $f(x,y,z) = xy$ and $A = \{(x,y,z) : x \in (0,1), y \in (0,1), 0 \leq z \leq 2 - x - y\}$. What is the value of $\int_A f(x,y,z)dxdydz$?

- ■ 1/6
- ☐ 0
- ☐ 1/4
- ☐ 1

Question 3 Let $f(x) = 2x - 1$ and $\varphi \in C_c^0(\mathbb{R})$ be a continuous function with compact support and $\int_{\mathbb{R}} \varphi(x)dx = 1$. Let $g(x) = \int_{\mathbb{R}} f(x-y)\varphi(y)dy$.
Which of the following statements is **true**?

- ☐ $g(x) = +\infty$ for any $x \in \mathbb{R}$
- ☐ $g(x) = 1$ for any $x \in \mathbb{R}$
- ■ There exists a constant $c \in \mathbb{R}$ depending on φ such that $g(x) = 2x + c$ for any $x \in \mathbb{R}$
- ☐ There exists a constant $c \in \mathbb{R}$ depending on φ such that $g(x) = cx - 1$

Question 4 Let $f \in C_c^0(\mathbb{R})$ and define $f_n(x) = nf(nx)$ for any $n \in \mathbb{N}$.
Which of the following statements is **true**?

- ■ $\lim_{n\to+\infty} \|f_n\|_{L^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})}$
- ☐ $\lim_{n\to+\infty} \|f_n - f\|_{L^1(\mathbb{R})} = 0$
- ☐ $\lim_{n\to+\infty} \|f_n\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$
- ☐ $\lim_{n\to+\infty} \|f_n\|_{L^2(\mathbb{R})} = 0$

Question 5 Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable functions and V be the Vitali set. Which of the following statements is **true**?

- ■ Every increasing function is measurable
- ☐ Every bounded function is measurable
- ☐ $\mathbb{1}_V$ is measurable
- ☐ $f \circ g$ is measurable

Second part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 6: *This question is worth 9 points.*

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Let $p \in [1, \infty)$.

(a) *(2 points).* State the dominated convergence theorem.

(b) *(4 points).* Prove that for any Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^p(0,1)$ there exist a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f, F \in L^p(0,1)$ such that $|f_{n_k}(x)| \leq F(x)$ for a.e. $x \in (0,1)$ and $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for a.e. $x \in (0,1)$.

(c) *(3 points)* Prove that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p((0,1))$ and $f \in L^p((0,1))$ such that the following two properties hold:

$$\|f_n - f\|_{L^p} \to 0 \qquad \text{as } n \to \infty$$

and there are no $F \in L^p(0,1)$ such that $f_n(x) \leq F(x)$ for any $n \in \mathbb{N}$ and for a.e. $x \in (0,1)$.

Hint: For (c), you may consider the functions $f_n = n^{\frac{1}{2p}} 1_{I_n}$, and $I_{2^m+i} = [i/2^m, (i+1)/2^m]$ for every $m \in \mathbb{N}$, $i \in 0, \dots, 2^m - 1$, namely $I_1 = [0,1]$, $I_2 = [0,1/2]$, $I_3 = [1/2,1]$, $I_4 = [0,1/4]$, $I_5 = [1/4,2/4]$...

Proof

(a) State the theorem as in class.

(b) For any $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ such that

$$\|f_m - f_j\|_{L^p} \leq 2^{-k}$$

for any $m, j \geq n_k$. We select the subsequence $\{f_{n_k}\}_{n_k}$. We define the function F defined as

$$F(x) = |f_{n_1}(x)| + \sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)|$$

We observe also that

$$|f_{n_k}(x)| = |f_{n_1}(x) + \sum_{j=1}^{k-1} f_{n_{j+1}} - f_{n_j}(x)| \leq |f_{n_1}(x)| + \sum_{j=1}^{k-1} |f_{n_{j+1}} - f_{n_j}(x)|$$

for any $k \geq 2$, therefore, for any $k \in \mathbb{N}$ we have that

$$|f_{n_k}(x)| \leq F(x)$$

for any $x \in (0,1)$. Finally we observe that $F \in L^p$ because

$$\|F\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

To prove the pointwise limit we use that $F \in L^p$ which implies that

$$\sum_{k=0}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| < \infty$$

for a.e. $x \in (0,1)$ and therefore for a.e. $x \in (0,1)$ we have $\{f_{n_k}(x)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, which implies that is a converging sequence and we call the limit $f(x)$.

(c) For any $n \in \mathbb{N}$ we observe that there exists unique $m \in \mathbb{N}$ and $i \in \{0, ..., 2^m - 1\}$ such that $n = 2^m + i$. We easily notice that $m(I_n) = 2^{-m}$ and $n^{1/2p} \leq 2^{\frac{m+1}{2p}}$ and therefore

$$\|f_n\|_{L^p}^p = 2^{\frac{m+1}{2}} \mathcal{L}(I_n) \leq 2^{\frac{1-m}{2}} \to 0 \,,$$

where the last holds when $n \to \infty$ (notice that if $n \to \infty$ then $m \to \infty$).

We now show that for any $x \in (0,1)$ and for any $M > 0$ there exists $n \in \mathbb{N}$ such that

$$|f_n(x)| \geq M \,,$$

which will conclude the proof.

We take $m \in \mathbb{N}$ such that $2^{\frac{m}{2p}} \geq M$. Since for any $m \in \mathbb{N}$ the family $\{I_{2^m}, I_{2^m+1}, \ldots, I_{2^{m+1}-1}\}$ is disjointed and covers $[0,1)$, there exists a natural number $n \in [2^m, 2^{m+1} - 1]$ such that $x \in I_n$. Therefore

$$f_n(x) \geq 2^{\frac{m}{2p}} \geq M \,.$$

Question 7: *This question is worth 8 points.*

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Let
$$f_n(x) = \frac{n}{x}\sin\left(\frac{x}{n}\right)e^{-x}, \quad x > 0$$

(a) *(1 point).* For every $x > 0$ prove that $\lim_{n\to\infty} f_n(x) = e^{-x}$.

(b) *(3 points).* Compute $\lim_{n\to\infty}\int_0^\infty f_n(x)dx$.

(c) *(4 points).* Using $\sin(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ rewrite $f_2(x)$ as a series and show that

$$\int_0^\infty f_2(x)dx = \sum_{n=0}^\infty \int_0^\infty \frac{(-1)^n x^{2n}}{4^n[(2n+1)!]}e^{-x}dx = \sum_{n=0}^\infty \frac{(-1)^n}{4^n(2n+1)}.$$

Hint: For point (c). To rigorously justify the first equality it may be useful to recall that $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ for any $x \in \mathbb{R}$.

Proof

(a) Thanks to the Taylor expansion of the sinus we have
$$\lim_{y\to 0}\frac{\sin(y)}{y} = 1,$$
therefore, taking $y = \frac{x}{n}$ and computing the limit as $n \to \infty$ we have
$$\lim_{n\to\infty} f_n(x) = e^{-x}.$$

(b) Using that $\sin(y)/y \le 1$ for any $y > 0$ we have $|f_n(x)| \le |e^{-x}| \in L^1(0,+\infty)$ for any $n \ge 1$. We apply the dominated convergence theorem, which enables to pass the limit under the integral, therefore we get $\lim_{n\to+\infty}\int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to+\infty} f_n(x)dx = 1$.

(c)
$$f_2(x) = \frac{2}{x}\sin(x/2)e^{-x} = \frac{2}{x}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!}e^{-x}$$

$$\begin{aligned}\int_0^{+\infty} f_2(x)dx &= \int_0^{+\infty}\frac{2}{x}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!}e^{-x}dx\\ &= \int_0^{+\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n}}{4^n(2n+1)!}e^{-x}dx\\ &\overset{\star}{=} \sum_{n=0}^{+\infty}\int_0^{+\infty}\frac{(-1)^n x^{2n}}{4^n(2n+1)!}e^{-x}dx\\ &= \sum_{n=0}^{+\infty} I_n\end{aligned}$$

In $\star$, we exploited the dominated convergence theorem for the sequence $g_N(x) = \sum_{n=0}^N \frac{(-1)^n x^{2n}}{2^{2n}(2n+1)!}e^{-x}$ which is such that $|g_N(x)| \le \sum_{j=0}^{2N}\frac{x^j}{2^j(j+1)!}e^{-x} \le \sum_{j=0}^{2N}\frac{x^j}{2^j j!}e^{-x} \le e^{-x/2} \in L^1$, where we used that $\sum_{j=0}^\infty \frac{x^j}{2^j j!} = e^{x/2}$.

We now compute each factor I_n:

$$\begin{aligned}
I_n &= \int_0^{+\infty} \frac{(-1)^n x^{2n}}{4^n(2n+1)!} e^{-x} dx \\
&= \frac{(-1)^n}{4^n(2n+1)!} \int_0^{+\infty} x^{2n} e^{-x} dx \\
&= \frac{(-1)^n}{4^n(2n+1)!} \left\{ \underbrace{\left[-e^{-x} x^{2n}\right]_0^{+\infty}}_{0} + \int_0^{+\infty} 2n e^{-x} x^{2n-1} dx \right\} \\
&= \frac{(-1)^n}{4^n(2n+1)!} \left\{ \underbrace{\left[-e^{-x} 2n x^{2n-1}\right]_0^{+\infty}}_{0} + \int_0^{+\infty} 2n(2n-1) e^{-x} x^{2n-2} dx \right\} \\
&= \dots \\
&= \frac{(-1)^n}{4^n(2n+1)!} \int_0^{+\infty} e^{-x} (2n)! dx = \frac{(-1)^n}{4^n(2n+1)}
\end{aligned}$$