Correction

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Which of the following statements is **true**?

- ■ The set of points where the function $\mathbb{1}_{[0,+\infty)}$ is not continous has measure zero.
- □ The function $\mathbb{1}_{[0,+\infty)}$ is a.e. equal to a continuous function.
- □ There exists a simple function that coincides a.e. with a continuous non-constant function.
- □ If $f : \mathbb{R} \to \mathbb{R}$ is a.e. equal to a continuous function, then f is a.e. continuous.

Question 2 $f(x) = \log(x) \in L^p((0, 1))$ if and only if

- ■ $p \in [1, +\infty)$.
- □ $p \in (1, +\infty]$.
- □ $p \in (1, 2]$.
- □ $p = 1$.

Question 3 Let A be the set of those numbers in $(0, 1)$ that have a decimal expansion containing at least one digit 7. Which of the following statements is **true**?

- □ A is not measurable.
- □ A is measurable and $0 < m(A) < 1$.
- □ A is measurable and $m(A) = 0$.
- ■ A is measurable and $m(A) = 1$.

Question 4 Consider the Fourier series $\sum_{n=1}^{+\infty} \frac{1}{n^3} \cos(nx)$. Which of the following statements is **true**?

- □ It is the Fourier series of a function $u \in C^3(\mathbb{R})$, but not in $C^\infty(\mathbb{R})$.
- ■ It is the Fourier series of a function $u \in C^1(\mathbb{R})$, but not in $C^3(\mathbb{R})$.
- □ It converges pointwise, but not uniformly, on $\mathbb{R}$.
- □ It cannot be the Fourier series of any $2\pi-$ periodic function in L^2.

Question 5 Let C denote the Cantor set. Which of the following statements is **true**?

- □ If $E \subseteq \mathbb{R}$ is a measurable set, then $m(E \cap [x, \infty)) \to 0$ as $x \to +\infty$.
- □ For any $E \subseteq \mathbb{R}$, it holds $m^*(E) = m^*(\bar{E})$.
- □ If $E \subseteq \mathbb{R}$ is not measurable, then $E \cap C$ can be not measurable.
- ■ If $E \subseteq \mathbb{R}$ is not measurable, then $E \cap C^c$ can be not measurable.

Question 6 Let f_1 be a smooth, compactly supported function, $f_2(x) = e^{-x^2}$, $f_3(x) = \mathbb{1}_{[-1,1]}(x)$ for $x \in \mathbb{R}$. The Fourier transform belongs to L^1:

- □ for one of the three functions.
- □ for none of the three functions.
- □ for all of the three functions.
- ■ for two of the three functions.

Correction

Question 7 Let $u : \mathbb{R}^2 \to \mathbb{R}$ solving $\Delta u = xy$ and $v(x, y) = \frac{1}{2}u(2x, 2y) + 2x^2 - y^2$. Then, v solves:

- ☐ $\Delta v = 8xy - 2$.
- ☐ $\Delta v = 2xy + 2$.
- ☐ $\Delta v = 2xy - 2$.
- ■ $\Delta v = 8xy + 2$.

Question 8 What are the real Fourier coefficients $\{a_n\}$ and $\{b_n\}$ of $\operatorname{sign}(\sin(x))$?

- ☐ $a_n = 0,\ b_{2n} = \frac{4}{n\pi},\ b_{2n+1} = 0$.
- ☐ $a_{2n} = \frac{4}{n\pi},\ a_{2n+1} = 0,\ b_{2n} = 0,\ b_{2n+1} = \frac{4}{(2n+1)\pi}$.
- ■ $a_n = 0,\ b_{2n} = 0,\ b_{2n+1} = \frac{4}{(2n+1)\pi}$.
- ☐ $a_{2n} = 0,\ a_{2n+1} = \frac{4}{n\pi},\ b_n = 0$.

Question 9 Let $F(x) = \left|\sum_{n=0}^{4} e^{2\pi i n x}\right|^2$. Which of the following statements is **true**?

- ■ $\int_0^1 F(x)dx = 5$.
- ☐ $\forall \varepsilon > 0 \quad \exists \delta < 1/2$ such that $|F(x)| \leq \varepsilon$ for all $x \in [\delta, 1 - \delta]$.
- ☐ F is not a trigonometric polynomial.
- ☐ $\int_0^1 F(x)^2 dx = 25$.

Question 10 Let f, g be two $1-$periodic functions in $L^2([0, 1])$. Assume that $f * g = 0$. Then:

- ■ f, g are orthogonal in $L^2([0, 1])$, namely $\int_0^1 fg = 0$.
- ☐ Either $f = 0$ or $g = 0$.
- ☐ f and g have the same L^2- norm, namely $\|f\|_{L^2} = \|g\|_{L^2}$.
- ☐ $fg = 0$.

Exam Exercises

Analysis IV

June 28, 2024

1 Solutions

1.1 E1 (Fatou's Lemma - 6 points)

1. (3 points). State and prove Fatou's lemma.
2. (1 point). Give an example of a sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n : [0,1] \to \mathbb{R}$, satisfying the assumptions of Fatou's lemma and such that *strict inequality* occurs.
3. (2 points). Let f be a non-decreasing function on $[0,1]$ and assume that for a.e. $x \in [0,1]$ there exists the following limit:

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}.$$

Prove that

$$\int_0^1 f'(x)\,\mathrm{d}x \leqslant f(1) - f(0).$$

Solution:

1. See Lecture Notes.
2. Here is an example of correct solution: let us define the sequence $\{f_n\}_{n\in\mathbb{N}}$ as follows:

$$f_{2n-1}(x) = \mathbb{1}_{[0,1/2]}(x),$$
$$f_{2n}(x) = \mathbb{1}_{(1/2,1)}(x).$$

Then

$$\liminf_{n\to\infty} f_n(x) = 0, \quad \text{for all } x \in \mathbb{R},$$

but

$$\int_0^1 f_n(x)\,\mathrm{d}x = 1, \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$0 = \int_0^1 \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x < \liminf_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 1.$$

Many students proposed an example on $\mathbb{R}^n$; points were not given in this case.

3. First, we extend the definition of f by letting $f(x) = f(1)$ for $x > 1$. Since f is non-decreasing, it is differentiable almost everywhere and, for almost every x, the representation

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

holds. Since f is non-decreasing, we note that its difference quotient $\frac{f(x+h)-f(x)}{h}$ is non-negative for every x and every h. Thus, by Fatou's lemma, we deduce

$$\begin{aligned}\int_0^1 f'(x)\,\mathrm{d}x &= \int_0^1 \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}\,\mathrm{d}x \leqslant \liminf_{h\to 0^+} \int_0^1 \frac{f(x+h)-f(x)}{h}\,\mathrm{d}x \\ &= \liminf_{h\to 0^+} \frac{1}{h}\int_1^{1+h} f(x)\,\mathrm{d}x - \frac{1}{h}\int_0^h f(x)\,\mathrm{d}x \\ &\leqslant f(1) - f(0)\end{aligned}$$

where we used the fact that f is non-decreasing again in the last inequality.

Note that the dominated convergence theorem could not be applied here to swap limit and integral.

1.2 E2 (Dominated Convergence - 4 points)

(4 points). Let $\{f_n\}_{n=1}^{+\infty}$ be a sequence of functions defined as follows:

$$f_n(x) = \begin{cases} \frac{n\sqrt{x}}{1+n^2x^2}, & x \in [0,1), \\ e^{-x^2/n}, & x \in [1,+\infty). \end{cases}$$

Compute $\lim_{n\to+\infty} f_n(x)$ for all $x \in [0,+\infty)$,

$$\lim_{n\to+\infty} \int_1^{+\infty} f_n(x)\,\mathrm{d}x \quad \text{and} \quad \lim_{n\to+\infty} \int_0^1 f_n(x)\,\mathrm{d}x$$

Solution:

1. The sequence of functions $\{f_n\}_{n=1}^{+\infty}$ converges pointwise to

$$f(x) = \begin{cases} 0, & x \in [0,1), \\ 1, & x \in [1,+\infty). \end{cases}$$

2. In $[0,1)$, f_n is dominated by $g(x) = \frac{1}{\sqrt{x}}$. Indeed, we have that

$$nx \le 1 + n^2x^2 \Leftrightarrow (1-nx)^2 + nx \ge 0$$

which is true for all $x \in [0,1]$. Since g is Lebesgue integrable on $[0,1]$, we deduce by Lebegue's dominated convergence theorem and the pointwise convergence to 0 that:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = 0.$$

In $[1,+\infty)$, we have $0 \le f_n \le f_{n+1}$ because

$$\frac{x^2}{n+1} \le \frac{x^2}{n} \Rightarrow -\frac{x^2}{n} \le -\frac{x^2}{n+1}.$$

By the monotone convergence theorem, we conclude

$$\lim_{n\to+\infty} \int_1^{+\infty} f_n(x)\,\mathrm{d}x = \int_1^{+\infty} 1\,\mathrm{d}x = +\infty.$$

Note that we cannot apply dominated convergence with dominant e^{-x^2}, because it does not dominate the sequence.

1.3 E3 (Fubini's Theorem - 5 points)

Let $f : (0,1) \to \mathbb{R}$ be a measurable function and let $1 \le p < +\infty$.

1. (1 point). Prove that, for all $y > 0$, we have

$$y^p m(\{x \in (0,1) : |f(x)| \geqslant y\}) \leqslant \int_0^1 |f(x)|^p\,\mathrm{d}x\,.$$

2. (2 points). Show that

$$\int_0^1 |f(x)|^p\,\mathrm{d}x = p\int_0^\infty y^{p-1} m(\{x \in (0,1) : |f(x)| \geqslant y\})\,\mathrm{d}y\,.$$

Hint: Notice that $|f(x)|^p = \int_0^{|f(x)|} py^{p-1}\,\mathrm{d}y$.

3. (2 points). Let $1 \le q < p < +\infty$ and assume that f satisfies

$$y^p m(\{x \in (0,1) : |f(x)| \geqslant y\}) \leqslant 1, \quad \text{for all } y \in \mathbb{R}_+. \tag{1}$$

Show that $f \in L^q((0,1))$.

4. (1 point). Consider $f(x) = x^{-1/p}$, for $1 \le p < +\infty$. Show that f satisfies (1), but $f \notin L^p((0,1))$.

Solution:

1. Recalling Chebyshev's inequality, i.e. considering the integral of the simple function $y 1_{\{x:\,|f(x)|\geqslant y\}} \le f$, we have

$$m(\{x \in (0,1) : |f(x)| \geqslant y\}) \le \frac{1}{y^p} \int_{|f|\geqslant y} |f(x)|^p \,\mathrm{d}x.$$

We deduce

$$y^p m(\{x \in (0,1) : |f(x)| \geqslant y\}) \leqslant \int_{|f|\geqslant y} |f(x)|^p \,\mathrm{d}x \leqslant \int_0^1 |f(x)|^p \,\mathrm{d}x.$$

2. Noticing that $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} \,\mathrm{d}y$ and using Tonelli's theorem, we deduce

$$\begin{aligned}\int_0^1 |f(x)|^p \,\mathrm{d}x &= \int_0^1 \left(\int_0^{|f(x)|} p\, y^{p-1} \,\mathrm{d}y \right) \mathrm{d}x = p \int_0^1 \left(\int_{\mathbb{R}} y^{p-1} \mathbb{1}_{[0,|f(x)|]}(y) \,\mathrm{d}y \right) \mathrm{d}x \\ &= p \int_{\mathbb{R}} \left(\int_0^1 \mathbb{1}_{\{(x,y)\in\mathbb{R}^2:\ 0\leqslant y\leqslant |f(x)|\}}(x,y) \,\mathrm{d}x \right) y^{p-1} \,\mathrm{d}y \\ &= p \int_{\mathbb{R}} m(\{x \in (0,1) : |f(x)| \geqslant y\}) \mathbb{1}_{[0,+\infty)}(y) y^{p-1} \,\mathrm{d}y \\ &= p \int_0^\infty y^{p-1} m(\{x \in (0,1) : |f(x)| \geqslant y\}) \,\mathrm{d}y.\end{aligned}$$

3. From the previous point, we have the representation

$$\int_0^1 |f(x)|^q \,\mathrm{d}x = q \int_0^\infty y^{q-1} m(\{x \in (0,1) : |f(x)| \geqslant y\}) \,\mathrm{d}y.$$

We compute

$$\begin{aligned}\int_0^\infty y^{q-1} m\left(\{x \in (0,1) : |f(x)| \geqslant y\}\right) \mathrm{d}y &= \int_0^1 y^{q-1} m\left(\{x \in (0,1) : |f(x)| \geqslant y\}\right) \mathrm{d}y \\ &\quad + \int_1^\infty y^{q-1} m\left(\{x \in (0,1) : |f(x)| \geqslant y\}\right) \mathrm{d}y \\ &\leqslant \int_0^1 y^{q-1} \mathrm{d}y + \int_1^\infty y^{q-1} y^{-p} \,\mathrm{d}y \\ &= \frac{y^q}{q}\bigg|_0^1 + \frac{y^{q-p}}{q-p}\bigg|_1^\infty \\ &= \frac{1}{q} + \frac{1}{p-q} = \frac{p}{q(p-q)} < +\infty.\end{aligned}$$

We stress that we used $q < p$ in the chain of inequalities above.

Note that the integral must be split, else we cannot prove it is finite because the conditions on integrability are different at zero and at infinity.
Moreover, you cannot prove that $f \in L^p$ (see point 3 for the counterexample).

4. For all $p \ge 1$, we have that $f \notin L^p((0,1))$, because $|f|^p = x^{-1} \notin L^1((0,1))$. On the other hand,

$$m(\{x \in (0,1) : x^{-1/p} \ge y\}) = m((0, y^{-p})) = y^{-p}.$$

Hence, $y^p m(\{x \in (0,1) : x^{-1} \ge y\}) = y^{p-p} = 1$.

1.4 E4 (Fourier Series - 5 points)

Consider the $2\pi-$ periodic function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \frac{e^x + e^{-x}}{2} \qquad \text{for } x \in (-\pi, \pi],$$

1. (1 point). Compute the norm $\|f\|_{L^2((-\pi,\pi])}$.
2. (1 point). Determine for which values of $x \in (-\pi, \pi]$ the Fourier Series converges pointwise to $f(x)$.
3. (2 points). Compute the Fourier series of f.
4. (1 point). Compute the value of
$$\sum_{k=0}^{+\infty} \frac{1}{(1+k^2)^2}.$$

Solution:

1. We have that
$$\|f\|_{L^2}^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{4}\int_{-\pi}^{\pi} (e^{2x} + e^{-2x} + 2)dx = \frac{1}{4}(e^{2\pi} - e^{-2\pi}) + \pi = \frac{\sinh(2\pi)}{2} + \pi.$$
Hence, $\|f\|_{L^2} = \sqrt{\frac{\sinh(2\pi)}{2} + \pi}$.
2. We have that
$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x = \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} e^{x(1-in)}dx + \int_{-\pi}^{\pi} e^{x(-1-in)}dx\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{1-in}\sinh(\pi - in\pi) + \frac{1}{1+in}\sinh(\pi + in\pi)\right) \\ &= \frac{1}{\pi(1+n^2)}\left(\sinh(\pi - in\pi)(1+in) + \sinh(\pi + in\pi)(1-in)\right) \\ &= \frac{1}{\pi(1+n^2)}\left(n\cosh(\pi)\sin(n\pi) + \cos(n\pi)\sinh(\pi)\right) \\ &= \frac{(-1)^n}{\pi(1+n^2)}\sinh(\pi). \end{aligned}$$
Hence, $f(x) = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\pi(1+n^2)}\sinh(\pi)e^{inx}$.
In many cases, you forgot the square root in the norm.
3. By Dirichlet's Theorem, the pointwise convergence holds for all $x \in [-\pi, \pi]$.
4. We have that $f(x) = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\pi(1+n^2)}\sinh(\pi)e^{inx}$. Hence, by Parseval's identity,
$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx \quad \Longrightarrow \quad \frac{\sinh(2\pi)}{4\pi} + \frac{1}{2} = \sum_{n\in\mathbb{Z}} |c_n|^2.$$
In many cases, you forgot the multiplicative constant in Parseval's identity.
Now, define $S := \sum_{n=1}^{+\infty} \frac{1}{(1+n^2)^2}$. It solves the equation $2S + 1 = \frac{\pi^2}{\sinh(\pi)^2}\left(\frac{\sinh(2\pi)}{4\pi} + \frac{1}{2}\right)$. Hence,
$$\sum_{n=0}^{+\infty} \frac{1}{(1+n^2)^2} = S + 1 = \frac{\pi^2}{2\sinh(\pi)^2}\left(\frac{\sinh(2\pi)}{4\pi} + \frac{1}{2}\right) + \frac{1}{2}.$$

1.5 E5 (Fourier Coefficients of Translation - 5 points)

Let f be a 1-periodic function on $\mathbb{R}$ such that $f\big|_{[0,1]}$ belongs to $L^2((0,1))$. Let c_n be the complex Fourier coefficients of f. For every $h \in \mathbb{R}$, let us define the function f_h by $f_h(x) = f(x-h)$.

1. (1 point). Give the Fourier expansion of f_h.
2. (1 point). Find all C^1 functions that are 1-periodic and satisfy $f'(x) = f(x-1/2)$ for all $x \in \mathbb{R}$.
3. (3 points). Prove that

$$\liminf_{h\to 0} \frac{\|f_h - f\|_{L^2((0,1))}}{|h|} > 0,$$

unless f is constant almost everywhere.

Solution:

1. We have

$$f_h(x) = f(x-h) = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi in(x-h)} = \sum_{n=-\infty}^{+\infty} c_n e^{-2\pi inh} e^{2\pi inx}.$$

2. The Fourier coefficients of f satisfy

$$(2\pi in - e^{\pi in})c_n = 0,$$

so $c_n = 0$ for all $n \in \mathbb{N}$, which implies $f = 0$.

You cannot use Fourier transform, because the periodic function is not $L^1(\mathbb{R})$.

3. Using Part 1 and Parseval's identity, we compute

$$\|f_h - f\|_{L^2((0,1))} = \left(\sum_{n=-\infty}^{+\infty} \left|c_n e^{-2\pi inh} - c_n\right|^2\right)^{\frac{1}{2}} = \left(\sum_{n=-\infty}^{+\infty} (e^{-2\pi inh}-1)^2 |c_n|^2\right)^{\frac{1}{2}} = \left(\sum_{n=-\infty}^{+\infty} 4\sin(\pi hn)^2 |c_n|^2\right)^{\frac{1}{2}}.$$

If f is constant almost everywhere, then $c_n = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$. As a consequence, we have $f_h = f$ and, thus, $\|f_h - f\|_{L^2((0,2\pi))} = 0$.

If f is not constant almost everywhere, then there exists $n \neq 0$ such that $c_n \neq 0$.

Then we compute

$$\liminf_{h\to 0} \frac{\|f_h - f\|_{L^2((0,1))}}{|h|} \geqslant \lim_{h\to 0} \frac{2|\sin(\pi hn)|\,|c_n|}{|h|} = 2\pi n|c_n| > 0.$$

Using Fatou's lemma did not enable you to reach the conclusion. Moreover, the existence of f' is not guaranteed by the available hypothesis.

1.6 E6 (PDE with Fourier (vanishing viscosity version) - 8 points)

Let $g \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$. Let us consider the following Cauchy problems:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x u^\varepsilon(t,x) - \varepsilon \partial^2_{xx} u^\varepsilon(t,x) = 0, & t>0,\ x\in\mathbb{R}, \\ u^\varepsilon(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{CP$_\varepsilon$}$$

and

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) = 0, & t>0,\ x\in\mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{CP}$$

1. (2 point). For a fixed $\varepsilon > 0$, write the formal solution u^ε of (CP$_\varepsilon$).
2. (1 point). Use the Fourier transform to write a formula for the solution u of (CP). In particular, observe that $u(t,\cdot)$ is given by a suitable translation of g.

3. (2 points). Prove that, for $\varepsilon > 0$,

$$\|u^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leq \|g\|_{L^2(\mathbb{R})} \quad \text{for all } t \geqslant 0.$$

4. (1 point). Show that $u^\varepsilon \in C^\infty((0,+\infty)\times\mathbb{R})$.

5. (2 points). Prove that the solutions u^ε of (CP$_\varepsilon$) converge to the solution u of (CP) in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, namely, for every $t > 0$,

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Solution:

1. By applying the Fourier transform to the problem (CP$_\varepsilon$), we have

$$\begin{cases} \partial_t \hat{u}^\varepsilon(t,\xi) + (2\pi i\,\xi + \varepsilon 4\pi^2|\xi|^2)\hat{u}^\varepsilon(t,\xi) = 0, \quad t > 0, \\ \hat{u}^\varepsilon(0,\xi) = \hat{g}(\xi), \end{cases} \tag{2}$$

where $\hat{u}^\varepsilon$ and $\hat{g}$ denote the Fourier transforms of u^ε and g with respect to the x-variable, respectively.

Separation of variables was not applicable here.

Solving (2), we get

$$\hat{u}^\varepsilon(t,\xi) = \hat{g}(\xi)e^{-(2\pi i\,\xi + 4\pi^2\varepsilon|\xi|^2)t}, \quad t \geq 0,\ \xi \in \mathbb{R}. \tag{3}$$

Recalling that $g \in \mathcal{S}(\mathbb{R})$ the properties of the Gauss–Weierstrass kernel H_t, the behavior of the Fourier transform with respect to shifting, and the convolution theorem, we apply the inverse Fourier transform to deduce

$$\begin{aligned} u^\varepsilon(t,x) = \mathcal{F}^{-1}[\hat{u}^\varepsilon(t,\xi)](x) &= \mathcal{F}^{-1}\left[\hat{g}(\xi)\, e^{-(2\pi i\,\xi + 4\pi^2\varepsilon|\xi|^2)t}\right](x) \\ &= \frac{1}{\sqrt{4\pi\varepsilon t}}\int_{\mathbb{R}} e^{-\frac{|x-t-y|^2}{4\varepsilon t}} g(y)\,\mathrm{d}y \\ &= g(x) * H_t^\varepsilon(x-t). \end{aligned}$$

Not recognizing the tranlsation was not penalized, but then we removed points for not justifying Plancherel rigorously.

2. By applying the Fourier transform to the problem (CP), we have

$$\begin{cases} \partial_t \hat{u}(t,\xi) + 2\pi i\,\xi\,\hat{u}(t,\xi) = 0, \quad t > 0, \\ \hat{u}(0,\xi) = \hat{g}(\xi), \end{cases} \tag{4}$$

where $\hat{u}$ and $\hat{g}$ denote the Fourier transforms of u and g with respect to the x-variable, respectively. Here, we use the notation

$$\begin{aligned} \mathcal{F}[f](\xi) = \hat{f}(\xi) &= \int_{\mathbb{R}} f(x)e^{-2\pi i x\,\xi}\,\mathrm{d}x, \\ \mathcal{F}^{-1}[f](x) = \check{f}(x) &= \int_{\mathbb{R}} f(\xi)e^{2\pi i x\,\xi}\,\mathrm{d}\xi, \end{aligned}$$

for the Fourier transform of a function f and its inverse.

Solving (4), we get

$$\hat{u}(t,\xi) = \hat{g}(\xi)e^{-2\pi i\xi t}, \quad t \geq 0,\ \xi \in \mathbb{R}. \tag{5}$$

Since $g \in \mathcal{S}(\mathbb{R})$, recalling the behavior of the Fourier transform with respect to shifts, we obtain

$$u(t,x) = \mathcal{F}^{-1}\left[\hat{g}(\xi)e^{-2\pi i\xi t}\right](x) = \int_{-\infty}^{\infty} \hat{g}(x)e^{-2\pi i\xi(x-t)}\,\mathrm{d}\xi = g(x-t).$$

3. We observe that, for all $t \geq 0$, $u^\varepsilon(t,\cdot) = g(x) * H_t^\varepsilon(\cdot - t)$ belongs to $L^2(\mathbb{R})$. Indeed,

$$\begin{aligned}\|u^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} &= \|g * H_t^\varepsilon(\cdot - t)\|_{L^2(\mathbb{R})} \\ &\leq \|g\|_{L^2(\mathbb{R})}\|H_t^\varepsilon(\cdot - t)\|_{L^1(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})},\end{aligned}$$

where we used that, in particular, $g \in L^2(\mathbb{R})$ and $\|H_t^\varepsilon(\cdot - t)\|_{L^1(\mathbb{R})} = 1$.
Alternatively, one can observe that $u^\varepsilon(t,\cdot) \in L^2(\mathbb{R})$ (because $H_t^\varepsilon(\cdot - t) \in \mathcal{S}(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ and then apply Parseval–Plancherel's identity:

$$\begin{aligned}\|u^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} &= \|\hat{u}^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \\ &= \left\| e^{-(2\pi i\,\cdot + 4\pi^2\varepsilon|\cdot|^2)t}\hat{g}\right\|_{L^2(\mathbb{R})} \\ &\leq \|\hat{g}\|_{L^2(\mathbb{R})} = \|g\|_{L^2(\mathbb{R})}\,.\end{aligned}$$

4. The result follows from Theorem 6.1 (i) of the Lecture Notes when observing that

$$u^\varepsilon(t,x) = (g * H_t^\varepsilon)(x - t)$$

is equal to a solution of the heat equation

$$\begin{cases}\partial_t u - \partial_{xx}^2 u = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R},\end{cases}$$

upon translation and rescaling.

5. From (3) and (5), we know that

$$\begin{aligned}\hat{u}^\varepsilon(t,\xi) &= \hat{g}(\xi)e^{-(2\pi i\,\xi + 4\pi^2\varepsilon|\xi|^2)t}, \\ \hat{u}(t,\xi) &= \hat{g}(\xi)e^{-2\pi i t\xi}.\end{aligned}$$

For any $t > 0$, we compute

$$\begin{aligned}\|\hat{u}^\varepsilon(t,\cdot) - \hat{u}(t,\cdot)\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left|\hat{g}(\xi)\left(e^{-(2\pi i\,\xi + 4\pi^2\varepsilon|\xi|^2)t} - e^{-it\xi}\right)\right|^2 \mathrm{d}\xi \\ &= \int_{\mathbb{R}} |\hat{g}(\xi)|^2 \left|e^{-4\pi^2\varepsilon|\xi|^2 t} - 1\right|^2 \mathrm{d}\xi.\end{aligned}$$

As $g \in \mathcal{S}(\mathbb{R})$ (and, in particular, it belongs to $L^2(\mathbb{R})$), using Lebesgue's dominated convergence theorem and then Parserval's equality, we deduce that

$$\|u^\varepsilon(t,\cdot) - v(t,\cdot)\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Many students found a dominant not in L^1.