Prof. : Maria Colombo
Analysis IV midterm - XXX
14.04.2022
Duration : 90 minutes

100

XXX-5

SCIPER : **FAKE-5**

Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 12 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A5 paper with notes handwritten by you personally.**
- For **multiple choice** questions, each question has **exactly one** correct answer. We give :

  +1 point if your answer is correct,
  0 points if you give no answer or more than one answer, or if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Part 1: multiple choice questions (5 points)

- For each question, mark the box corresponding to the correct answer.
- Each question has **exactly one** correct answer.
- For each multiple choice question, we give :

  +1 point if your answer is correct,
  0 points if you give no answer or more than one answer, or if your answer is incorrect.

Question 1 : Let $E \subseteq \mathbb{R}^d$ be a measurable set. Which of the following statements is true ?

- ☐ $m(E) + m^*(A) < m^*(E \cup A)$ for all set $A \subseteq \mathbb{R}^d$ such that $A \cap E = \emptyset$.
- ☐ $m(E) + m^*(A) = m^*(E \cup A)$ for all set $A \subseteq \mathbb{R}^d$ such that $A \cap E = \emptyset$.
- ☐ None of these answers is correct.
- ☐ $m(E) + m^*(A) > m^*(E \cup A)$ for all set $A \subseteq \mathbb{R}^d$ such that $A \cap E = \emptyset$.

Question 2 : Which of the following statements is true?

- ☐ If $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n\geq 1}$ is a sequence of measurable and nonnegative functions with $\int_{\mathbb{R}} f_n(x)dx \leq 1$ for all $n \geq 0$, and $f_n \to 0$ pointwise, then $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx \to 0$.
- ☐ If $\{f_n\}_{n\geq 1}$ is a sequence of nonnegative measurable functions, then $\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)dx$.
- ☐ If $\{f_n : \mathbb{R} \to [0, \infty)\}_{n\geq 1}$ is a sequence of measurable functions with $f_1 \geq f_2 \geq ...$, and f_n converges pointwise to a nonnegative measurable function f, then $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx$.
- ☐ If $\{f_n : \mathbb{R} \to [-1, 1]\}_{n\geq 1}$ is a sequence of measurable functions with $f_1 \leq f_2 \leq ...$, and f_n converges pointwise to an absolutely integrable function f, then $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx$.

Question 3 : For $n \geq 1$, let $f_n : \mathbb{R}^d \to \mathbb{R}$ be measurable. Which of the following statements is true ?

- ☐ If the sequence $\{f_n\}_n$ is a convergent sequence in $L^3(\mathbb{R}^d)$ and is bounded in $L^\infty(\mathbb{R}^d)$, then $\{f_n\}_n$ is a convergent sequence in $L^4(\mathbb{R}^d)$.
- ☐ If the sequence $\{f_n\}_n$ converges in $L^2(\mathbb{R}^d)$ and in $L^4(\mathbb{R}^d)$ to a function $f \in L^2(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$ then the sequence $\{f_n\}_n$ converges almost everywhere to f in Ω.
- ☐ If the sequence $\{f_n\}_n$ is bounded in $L^4(\mathbb{R}^d)$ then it is bounded in $L^3(\mathbb{R}^d)$.
- ☐ $\|f\|_{L^2(\mathbb{R}^d)}\|f\|_{L^4(\mathbb{R}^d)} \leq \|f\|_{L^3(\mathbb{R}^d)}$ for all $f \in L^3(\mathbb{R}^d)$.

Question 4 : Define the diameter of a set $A \subseteq \mathbb{R}$ as

$$\mathrm{diam}(A) = \sup_{x,y\in A} |x - y| .$$

Which of the following statements is true ?

- ☐ diam is subadditive, i.e., $\mathrm{diam}(A \cup B) \leq \mathrm{diam}(A) + \mathrm{diam}(B)$ for all sets $A, B \subseteq \mathbb{R}$.
- ☐ There is a measurable set $A \subseteq \mathbb{R}$ such that $\mathrm{diam}(A) < +\infty$ and $\mathrm{diam}(A) < m(A)$.
- ☐ If $A \subseteq \mathbb{R}$ is such that $\mathrm{diam}(A) < +\infty$ and there exist $x, y \in A$ with $\mathrm{diam}(A) = |x - y|$ (i.e., the sup is a max), then A is measurable.
- ☐ For every $r \geq \lambda \geq 0$, there is a measurable set $A \subseteq \mathbb{R}$ such that $m(A) = \lambda$ and $\mathrm{diam}(A) = r$.

Question 5 : For $n \geq 1$, define $f_n : [0,1] \to \mathbb{R}$ as

$$f_n(x) = g_n(x) + 2n \ln\left(1 + \frac{x^2}{n^2}\right) \text{ where } g_n(x) = \begin{cases} 2n & \text{if } x \in [0, 1/n], \\ 0 & \text{if } x \in (1/n, 1]. \end{cases}$$

Then $\lim_{n \to +\infty} \int_0^1 f_n(x)dx$ is equal to

- ☐ $+\infty$
- ☐ 0
- ☐ 2
- ☐ 1

Part 2: open questions (17 points)

- Please answer the question in the designated empty space below each exercise.
- If you don't have enough space, you may use the additional empty pages at the end of the exam. In this case, please **mark very clearly** (if possible by indicating the page numbers) i) on the page of the relevant exercise that you are continuing the solution elsewhere and ii) which exercise you are continuing on the additional pages.
- Your answer should be carefully justified. The derivation of the results must be clear and complete.
- Leave the check-boxes empty; they are used for grading only.

Exercise 1 (*4.5 points*)

☐0 ☐.5 ☐1 ☐.5 ☐2 ☐.5 ☐3 ☐.5 ☐4 ☐.5 *Do not write here.*

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$|g(x)| \leq |x|^2 \text{ for all } x \in \mathbb{R}.$$

i. *(2 points)* For $n \geq 1$, let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable functions which pointwise converge to $f : \mathbb{R} \to \mathbb{R}$. Furthermore assume that there is a function $F \in L^2(\mathbb{R})$ such that for every $n \geq 1$, $|f_n(x)| \leq F(x)$ for almost every $x \in \mathbb{R}$. Show that $g \circ f \in L^1(\mathbb{R})$ and

$$\lim_{n\to+\infty} \|g \circ f_n - g \circ f\|_{L^1(\mathbb{R})} = 0.$$

ii. *(2.5 points)* Let $\{f_n\}_n \subseteq L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$ be such that

$$\lim_{n\to+\infty} \|f_n - f\|_{L^2(\mathbb{R})} = 0.$$

Is it still true that

$$\lim_{n\to+\infty} \|g \circ f_n - g \circ f\|_{L^1(\mathbb{R})} = 0 \text{ ?}$$

Exercise 2 (*5.5 points*)

0 .5 1 .5 2 .5 3 .5 4 .5 5 .5 *Do not write here.*

i. (*1.5 points*) Compute

$$\lim_{n\to\infty} n^2 \int_0^{1/2} \frac{\tan(x)\sin(x/n^2)}{x} dx.$$

ii. (*1.5 points*) Compute

$$\lim_{n\to\infty} \int_0^{\infty} \frac{1}{1+x} \arctan(n/x^3)\, dx.$$

iii. (*2.5 points*) Prove the equality

$$\int_0^1 \frac{x^p \ln(x)}{x-1}\, dx = \sum_{k=1}^{\infty} \frac{1}{(p+k)^2}$$

for $p > -1$.

Hint : Expand a part of the integrand into a series and justify the term by term integration.

Exercise 3 (*7 points*)

0 .5 1 .5 2 .5 3 .5 4 .5 5 .5 6 .5 7 *Do not write here.*

i. (*1 point*) State in a complete form Fubini's theorem.

ii. (*3.5 points*) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} y^{-2} & \text{if } 0 < x < y < 1 \\ -x^{-2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

Compute both the expressions appearing in the thesis of Fubini's theorem which involve double integrals (in x and y) of f and argue why it is not a contradiction to Fubini's theorem.

iii. (*2.5 points*) Let $N > 0$. Prove Fubini's theorem for the function $f = \mathbf{1}_E$, where $E \subseteq [-N, N]^2$ is measurable. In this proof, you can assume (as also done in the proof presented in class) the measurability of the functions involved in your computations.

+100/10/3+