First part: multiple choice questions

For each question, mark the box corresponding to the correct answer.

Question 1 Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$, where $f_n(x) = \frac{1}{nx+1}$. Which of the following is **true**?

- [ ] It converges to 0 for every $x \in [0, \infty)$.
- [X] It converges to 0 in $L^2((0, \infty))$.
- [ ] It converges to 0 in $L^1((0, \infty))$.
- [ ] It converges to 0 uniformly in $[0, 1]$.

Question 2 For a given parameter $a > 0$, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f_a(x) = \frac{1}{|x|^a}.$$

Which of the following statements is **true**?

- [ ] $f_a \in L^p(\mathbb{R} \setminus (-1, 1))$ if and only if $p < 2/a$.
- [ ] $f_a \in L^2(\mathbb{R})$.
- [X] $f_a \notin L^p(\mathbb{R})$ for all $p \in [1, +\infty]$.
- [ ] $f_a \in L^p((-1, 1))$ if and only if $p > 1/a$.

Question 3 Let $\Gamma := \{(x, 2x) : x \in (0, 1)\} \subset \mathbb{R}^2$ and let m be the Lebesgue measure in $\mathbb{R}^2$. Then, $m(\Gamma)$ is equal to

- [ ] 2.
- [ ] 1.
- [ ] $\sqrt{5}$.
- [X] 0.

Question 4 Let $f : [0, 1] \to \mathbb{R}$. Which of the following is **true**?

- [ ] If $\{x \in [0, 1] : f(x) = c\}$ is measurable for every $c \in \mathbb{R}$, then f is measurable.
- [X] If f is continuous a.e., then f is measurable.
- [ ] If f is continuous a.e., then there exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that $f = g$ a.e..
- [ ] If f is continuous and $f = g$ a.e. for some $g : [0, 1] \to \mathbb{R}$, then g is continuous a.e..

Question 5 Let $A \subseteq \mathbb{R}^d$. Let $\overline{A}$ be the closure of A and $\text{int}(A)$ be the interior of A (i.e. the biggest open set contained in A). Which of the following is **true**?

- [ ] If A is open, then $m(A) = m(\overline{A})$.
- [ ] $\text{int}(A) = \emptyset$ if and only if $m^*(A) = 0$.
- [X] If $m(\text{int}(A)) = m(\overline{A}) < +\infty$, then A is measurable.
- [ ] There exists a measurable set $E \subset \mathbb{R}^d$ with $m(E) > 0$ such that $|x - y| \in \mathbb{Q}$ for all $x, y \in$ E

Analysis IV Exercises

April 9, 2025

1 Midterm 2025

1.1 Open questions

Exercise 1.1 (**6 points**). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1((0,1))$ which converges in L^1 to $f \in L^1((0,1))$.

1. (**4 points**). Prove that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a function $F \in L^1((0,1))$ such that $|f_{n_k}(x)| \le F(x)$ for a.e. $x \in (0,1)$ and every $k \in \mathbb{N}$.
2. (**2 points**). Let $\{g_n\}_{n\in\mathbb{N}} \subset L^1((0,1))$ be a sequence of functions that converge pointwise a.e. to g and such that $|g_n(x)| \le |f_n(x)|$ for a.e. $x \in (0,1)$ and every $n \in \mathbb{N}$. Prove that

$$\int_{(0,1)} g_n(x)dx \to \int_{(0,1)} g(x)dx.$$

Solution.

1. See Lecture Notes.
2. Suppose by contradiction that this is not the case. Then, there must exist a subsequence $n_k \uparrow \infty$ and some $\epsilon > 0$ for which

$$\left|\int_{(0,1)} g_{n_k}(x)dx - \int_{(0,1)} g(x)dx\right| \ge \epsilon \qquad \forall k \in \mathbb{N}. \tag{1.1}$$

By point 1. applied to the sequence $\{f_{n_k}\}_{k\in\mathbb{N}}$, we can extract a further subsequence $n_{k_j} \uparrow \infty$ such that $|g_{n_{k_j}}(x)| \le |f_{n_{k_j}}(x)| \le F(x)$ for every $j \in \mathbb{N}$ and almost every $x \in (0,1)$, where F is some function in $L^1((0,1))$. Then, from the dominated convergence theorem we deduce that

$$\lim_{j\to\infty} \int_{(0,1)} g_{n_{k_j}}(x)dx = \int_{(0,1)} g(x)dx,$$

which is in contradiction with (1.1) above.

□

Exercise 1.2 (**6 points**). Consider the set $E = \left\{(x,y) \in (0,\infty)\times\mathbb{R} : |y| < \frac{1}{x^4+x^6}\right\}$ and the function $f : E \to \mathbb{R}$ defined as $f(x,y) = x^7 y$.

1. (**2 points**). Prove that

$$\int_E |f| = \int_0^\infty \frac{x^7}{(x^4+x^6)^2}dx.$$

2. (**1 point**). Show that $f \notin L^\infty(E)$.
3. (**3 points**). Determine all the values of $p \in [1, \infty)$ for which $f \in L^p(E)$.

Solution.

1. Let us call $\ell(x) := 1/(x^4 + x^6)$. We can apply Tonelli's Theorem to the nonnegative function $|f|$ to get
$$\int_E |f(x,y)|dxdy = \int_0^{+\infty} \int_{-\ell(x)}^{\ell(x)} |f(x,y)|dydx = 2\int_0^{+\infty} \int_0^{\ell(x)} |f(x,y)|dydx,$$
where in the second equality we used the even symmetry of $|f(x,y)|$ in y. Now observe that, for any $x > 0$, we have
$$\int_0^{\ell(x)} |f(x,y)|dy = \int_0^{\ell(x)} x^7 y dy = \frac{1}{2}\ell(x)^2 x^7 = \frac{1}{2}\frac{x^7}{(x^4+x^6)^2}.$$
Plugging this equality in the formula above we conclude point 1.

2. To prove that $f \notin L^\infty(E)$ it is enough to show that $m(\{(x,y) \in E : f(x,y) > n\}) > 0$ for every $n \in \mathbb{N}$. We have,
$$\{(x,y) \in E : f(x,y) > n\} = \big\{(x,y) \in (0,\infty)^2 : nx^{-7} < y < (x^4+x^6)^{-1}\big\}$$
which contains, for example, the set
$$\big\{(x,y) \in (0,\infty)^2 : x > 3n, nx^{-7} < y < (x^4+x^6)^{-1}\big\}$$
that is nonempty since $3^{-7}n^{-6} < (3^4n^4 + 3^6n^6)^{-1}$. By using Tonelli again, we may compute
$$\begin{aligned} m(\{(x,y) \in E : f(x,y) > n\}) &\geq m(\big\{(x,y) \in (0,\infty)^2 : x > 3n, nx^{-7} < y < (x^4+x^6)^{-1}\big\}) \\ &= \int_{3n}^{\infty} \int_{nx^{-7}}^{\ell(x)} 1 dydx = \int_{3n}^{\infty} (\ell(x) - nx^{-7})dx > 0, \end{aligned}$$
because $\ell(x) > nx^{-7}$ for all $x > 3n$, as observed above.

3. Let $p \in [1,\infty)$. Proceeding analogously as in point 1., we apply Tonelli to the nonnegative function $|f|^p$ to get
$$\begin{aligned} \int_E |f(x,y)|^p dxdy &= \int_0^{\infty} \int_{-\ell(x)}^{\ell(x)} |f(x,y)|^p dydx = 2\int_0^{+\infty} \int_0^{\ell(x)} |f(x,y)|^p dydx \\ &= 2\int_0^{+\infty} \int_0^{\ell(x)} x^{7p} y^p dydx = \frac{2}{p+1} \int_0^{\infty} \frac{x^{7p}}{(x^4+x^6)^{p+1}} dx. \end{aligned}$$
Therefore, $f \in L^p(E)$ if and only if the function $h(x) := x^{7p}/(x^4+x^6)^{p+1}$ has finite integral in $(0,\infty)$. We split this integral in the two domains of integration $(0,1)$ and $(1,\infty)$ and treat the two quantities separately.
Firstly, observe that $x^4 \geq x^6$ in $(0,1)$, which implies that
$$\frac{x^{3p-4}}{2^{p+1}} \leq h(x) \leq x^{3p-4} \qquad \forall x \in (0,1).$$
Therefore, $h(x)$ is integrable in $(0,1)$ if and only if x^{3p-4} is integrable in $(0,1)$:
$$\int_0^1 h(x)dx < \infty \iff 3p-4 > -1 \iff p > 1.$$

Secondly, observe that $x^4 \leq x^6$ in $(1, \infty)$, which implies that

$$x^{p-6} \leq h(x) \leq \frac{x^{p-6}}{2^{p+1}} \qquad \forall x \in (1, \infty).$$

Therefore, $h(x)$ is integrable in $(1, \infty)$ if and only if x^{p-6} is integrable in $(1, \infty)$:

$$\int_1^\infty h(x)dx < \infty \Longleftrightarrow p - 6 < -1 \Longleftrightarrow p < 5.$$

Combining the two conditions we get that $f \in L^p(E)$ if and only if $1 < p < 5$.

$\square$

Exercise 1.3 (**6 points**). Let $\mathcal{B}_r(x) \subset \mathbb{R}^2$ be the open ball of radius $r > 0$ and center $x \in \mathbb{R}^2$ whose measure is $m(\mathcal{B}_r(x)) = \pi r^2$.

1. (**1 point**). Prove that there exists a positive constant $c_1 > 0$ such that, for every ball $\mathcal{B}_r(x) \subset \mathbb{R}^2$ one can find an open rectangle R that satisfies

$$\mathcal{B}_r(x) \subset R \qquad \text{and} \qquad m(R) \leq c_1 m(\mathcal{B}_r(x)).$$

2. (**2 points**). Prove that there exists a constant $c_2 > 0$ such that, for every open rectangle $R \subset \mathbb{R}^2$ one can find balls $\{\mathcal{B}_{r_n}(x_n)\}_{n=0}^N$ that satisfy

$$R \subset \bigcup_{n=0}^N \mathcal{B}_{r_n}(x_n) \qquad \text{and} \qquad \sum_{n=0}^N m(\mathcal{B}_{r_n}(x_n)) \leq c_2 m(R).$$

3. (**1 point**). For every set $E \subseteq \mathbb{R}^2$, define

$$\sigma^*(E) := \inf \left\{ \sum_{n=0}^\infty m(\mathcal{B}_{r_n}(x_n)) : \{\mathcal{B}_{r_n}(x_n)\}_{n\in\mathbb{N}} \text{ covering of } E \right\} \in [0, \infty].$$

Prove that $m^*(E) \leq c_1 \sigma^*(E)$ and $\sigma^*(E) \leq c_2 m^*(E)$.

4. (**2 points**). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$|F(x) - F(y)| \leq |x - y| \quad \forall x, y \in \mathbb{R}^2.$$

Prove that $m^*(F(E)) \leq c_1 c_2 m^*(E)$ for every $E \subset \mathbb{R}^2$.

Solution.

1. Let $x = (x_1, x_2)$; take $R = (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r)$. Then, clearly $\mathcal{B}_r(x) \subset R$ and

$$m(R) = 4r^2 = c_1 m(\mathcal{B}_r(x)),$$

where $c_1 = 4/\pi$.

2. Without loss of generality assume that $R = (0, \ell_1) \times (0, \ell_2)$, with $0 < \ell_1 \leq \ell_2 < \infty$. We choose

$$N = \left\lfloor \frac{\ell_2}{\ell_1} \right\rfloor, \quad r_n = r = 2\ell_1, \quad x_n = (0, n\ell_1).$$

By this choice, $R \subset \bigcup_{n=0}^N \mathcal{B}_r(x_n)$. Moreover,

$$\sum_{n=0}^N m\left(\mathcal{B}_r(x_n)\right) = (N+1)4\pi\ell_1^2 \leq 8\pi\ell_1\ell_2 = 8\pi m(R),$$

because the $N + 1$ balls have the same measure and $(N + 1)\ell_1 \leq (\ell_1 + \ell_2) \leq 2\ell_2$ by definition of N. Hence we have point 2. with $c_2 = 8\pi$.

3. We prove the first inequality, the second being analogous. Let $E \subseteq \mathbb{R}^2$, and let $\{\mathcal{B}_{r_n}(x_n)\}_{n\in\mathbb{N}}$ be a covering of E. Then, by point 1. we can find rectangles R_n with $\mathcal{B}_{r_n}(x_n) \subset R_n$ and $m(R_n) \leq c_1 m(\mathcal{B}_{r_n}(x_n))$. Then, by definition of outer measure, we have:

$$m^*(E) \leq \sum_{n\in\mathbb{N}} m(R_n) \leq c_1 \sum_{n\in\mathbb{N}} m(\mathcal{B}_{r_n}(x_n)).$$

   By the arbitrariness of the covering $\{\mathcal{B}_{r_n}(x_n)\}_{n\in\mathbb{N}}$ we deduce that $m^*(E) \leq c_1\sigma^*(E)$.

   To prove the other implication, it suffices to repeat the same argument using point 2. instead of 1.

4. First observe that for any ball $\mathcal{B}_r(x) \subset \mathbb{R}^2$, since F is a 1-Lipschitz function, we have

$$F(\mathcal{B}_r(x)) \subset \mathcal{B}_r(F(x)).$$

   Let now $E \subset \mathbb{R}^2$ be any set. Consider a covering $\{\mathcal{B}_{r_n}(x_n)\}_{n\in\mathbb{N}}$ of E. Then $\{\mathcal{B}_{r_n}(F(x_n))\}_{n\in\mathbb{N}}$ is a covering of $F(E)$. Hence,

$$\sigma^*(F(E)) \leq \sum_{n\in\mathbb{N}} m(\mathcal{B}_{r_n}(F(x_n))) = \sum_{n\in\mathbb{N}} m(\mathcal{B}_{r_n}(x_n)),$$

   and by the arbitrariness of the covering we deduce that

$$\sigma^*(F(E)) \leq \sigma^*(E).$$

   In order to conclude point 4. we use point 3. together with the latter inequality:

$$m^*(F(E)) \leq c_1\sigma^*(F(E)) \leq c_1\sigma^*(E) \leq c_1c_2m^*(E).$$

□