EPFL

Teacher: Maria Colombo
Midterm - MA
09/04/2025
90 minutes

X-5

Student 5

SCIPER : **XXXXX** Room : **XXX** Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 8 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A5 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  - +1 points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - 0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer.

Question 1 Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$, where $f_n(x) = \frac{1}{nx+1}$. Which of the following is **true**?

- ☐ It converges to 0 uniformly in $[0,1]$.
- ☐ It converges to 0 in $L^2((0,\infty))$.
- ☐ It converges to 0 for every $x \in [0,\infty)$.
- ☐ It converges to 0 in $L^1((0,\infty))$.

Question 2 For a given parameter $a > 0$, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f_a(x) = \frac{1}{|x|^a}.$$

Which of the following statements is **true**?

- ☐ $f_a \in L^p(\mathbb{R} \setminus (-1,1))$ if and only if $p < 2/a$.
- ☐ $f_a \notin L^p(\mathbb{R})$ for all $p \in [1,+\infty]$.
- ☐ $f_a \in L^2(\mathbb{R})$.
- ☐ $f_a \in L^p((-1,1))$ if and only if $p > 1/a$.

Question 3 Let $f : [0,1] \to \mathbb{R}$. Which of the following is **true**?

- ☐ If f is continuous a.e., then there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $f = g$ a.e..
- ☐ If $\{x \in [0,1] : f(x) = c\}$ is measurable for every $c \in \mathbb{R}$, then f is measurable.
- ☐ If f is continuous and $f = g$ a.e. for some $g : [0,1] \to \mathbb{R}$, then g is continuous a.e..
- ☐ If f is continuous a.e., then f is measurable.

Question 4 Let $A \subseteq \mathbb{R}^d$. Let $\overline{A}$ be the closure of A and $\text{int}(A)$ be the interior of A (i.e. the biggest open set contained in A). Which of the following is **true**?

- ☐ If $m(\text{int}(A)) = m(\overline{A}) < +\infty$, then A is measurable.
- ☐ $\text{int}(A) = \emptyset$ if and only if $m^*(A) = 0$.
- ☐ If A is open, then $m(A) = m(\overline{A})$.
- ☐ There exists a measurable set $E \subset \mathbb{R}^d$ with $m(E) > 0$ such that $|x - y| \in \mathbb{Q}$ for all $x, y \in A$.

Question 5 Let $\Gamma := \{(x, 2x) : x \in (0,1)\} \subset \mathbb{R}^2$ and let m be the Lebesgue measure in $\mathbb{R}^2$. Then, $m(\Gamma)$ is equal to

- ☐ 1.
- ☐ ∞.
- ☐ 2.
- ☐ 0.

Second part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 6: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1((0,1))$ which converges in L^1 to $f \in L^1((0,1))$.

(a) (**4 points**). Prove that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and a function $F \in L^1((0,1))$ such that $|f_{n_k}(x)| \leq F(x)$ for a.e. $x \in (0,1)$ and every $k \in \mathbb{N}$.

(b) (**2 points**). Let $\{g_n\}_{n\in\mathbb{N}} \subset L^1((0,1))$ be a sequence of functions that converge pointwise a.e. to g and such that $|g_n(x)| \leq |f_n(x)|$ for a.e. $x \in (0,1)$ and every $n \in \mathbb{N}$. Prove that

$$\int_0^1 g_n(x)dx \to \int_0^1 g(x)dx.$$

Question 7: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Consider the set $E = \left\{(x,y) \in (0,\infty) \times \mathbb{R} : |y| < \frac{1}{x^4+x^6}\right\}$ and the function $f : E \to \mathbb{R}$ defined as $f(x,y) = x^7 y$.

(a) (**2 points**). Prove that

$$\int_E |f| = \int_0^\infty \frac{x^7}{(x^4+x^6)^2}\,dx.$$

(b) (**1 point**). Show that $f \notin L^\infty(E)$.

(c) (**3 points**). Determine all the values of $p \in [1,\infty)$ for which $f \in L^p(E)$.

+124/8/17+

Question 8: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $\mathcal{B}_r(x) \subset \mathbb{R}^2$ be the open ball of radius $r > 0$ and center $x \in \mathbb{R}^2$, whose measure is $m(\mathcal{B}_r(x)) = \pi r^2$.

(a) (**1 point**). Prove that there exists a positive constant $c_1 > 0$ such that, for every ball $\mathcal{B}_r(x) \subset \mathbb{R}^2$, one can find an open box R that satisfies

$$\mathcal{B}_r(x) \subset R \qquad \text{and} \qquad m(R) \leq c_1 m(\mathcal{B}_r(x)).$$

(b) (**2 points**). Prove that there exists a constant $c_2 > 0$ such that, for every open box $R \subset \mathbb{R}^2$, one can find balls $\{\mathcal{B}_{r_n}(x_n)\}_{n=0}^N$ that satisfy

$$R \subset \bigcup_{n=0}^{N} \mathcal{B}_{r_n}(x_n) \qquad \text{and} \qquad \sum_{n=0}^{N} m(\mathcal{B}_{r_n}(x_n)) \leq c_2 m(R).$$

Hint: assume that the box has sides $L_1 \leq L_2$ and find a covering of balls with radius L_1.

(c) (**1 point**). For every set $E \subseteq \mathbb{R}^2$, define

$$\sigma^*(E) := \inf \left\{ \sum_{n=0}^{\infty} m(\mathcal{B}_{r_n}(x_n)) : \{\mathcal{B}_{r_n}(x_n)\}_{n\in\mathbb{N}} \text{ covering of } E \right\} \in [0, \infty].$$

Prove that $m^*(E) \leq c_1 \sigma^*(E)$ and $\sigma^*(E) \leq c_2 m^*(E)$.

(d) (**2 points**). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$|F(x) - F(y)| \leq |x - y| \quad \forall x, y \in \mathbb{R}^2.$$

Prove that $m^*(F(E)) \leq c_1 c_2 m^*(E)$ for every $E \subset \mathbb{R}^2$.