EPFL

Teacher : Maria Colombo
Analysis IV (MATH-205) - MA
17/04/2024
Duration : 90 minutes

95

Student 5

SCIPER : **999004**

Do not turn the page before the start of the exam. This document is double-sided, has 12 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A5 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  +1 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Consider the triangle $T \subset \mathbb{R}^2$ with vertices in $A = (0,0), B = (1,1), C = (1,-1)$. What is the value of $\int_T (x+y)^2$?

- ☐ 8/9
- ☐ 2/3
- ☐ It is not defined because $(x+y)^2$ is not absolutely integrable.
- ☐ 1/3

Question 2 Let $f(x) = \frac{e^{-x}}{\sqrt{x}}$. f belongs to $L^p(0,1)$ if and only if

- ☐ $p \in [1,2)$
- ☐ $p \in [2,+\infty)$
- ☐ $p = 2$
- ☐ $p = +\infty$

Question 3 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative functions in $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Which of the following statements is **true**?

- ☐ If $\liminf_{n\to\infty} \|f_n\|_{L^1} = 0$, then, for all $\varepsilon > 0$, we have $m(\{x \in \Omega : \liminf_{n\to\infty} f_n(x) < \varepsilon\}) = m(\Omega)$.
- ☐ If $\|f_n\|_{L^2} \to +\infty$, then $\|f_n\|_{L^1} \to +\infty$.
- ☐ If $\|f_n\|_{L^2} = C > 0$, then, for all $M > 0$, we have $m(\{x \in \mathbb{R}^d : |f_n(x)| \leq M\}) \leq \frac{C^2}{M^2}$.
- ☐ If $f_n \to 0$ almost everywhere, then $\|f_n\|_{L^1} \to 0$.

Question 4 Let $\alpha \in \mathbb{R}$. With Hölder's inequality, determine: the function $\frac{f(x)}{x^\alpha}$ belongs to $L^1(1,+\infty)$ for all $f \in L^2(1,+\infty)$ if and only if

- ☐ $\alpha \leq 0$
- ☐ $0 \leq \alpha \leq 1/2$
- ☐ $\alpha = 1/2$
- ☐ $\alpha > 1/2$

Question 5 Which of the following statements is **true**?

- ☐ Every function $F : \mathbb{R} \to \mathbb{R}$ that vanishes outside the Cantor set is measurable.
- ☐ Every function $F : \mathbb{R} \to \mathbb{R}$ that vanishes on the Cantor set is measurable.
- ☐ If the function $F : \mathbb{R} \to \mathbb{R}$ is measurable, then the set $\{x \in \mathbb{R} : F(x) = x\}$ can be not measurable.
- ☐ Every function $F : \mathbb{R} \to \{0\} \cup \{1\}$ is measurable.

Second part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 6: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Fix $0 < \gamma \leq 1/3$ and define $C_{0,\gamma} = [0,1]$. For every $n \in \mathbb{N}$, the set $C_{n,\gamma}$ is obtained from $C_{n-1,\gamma}$ by removing the open middle subinterval of length γ^n from each of the intervals in $C_{n-1,\gamma}$. For example, $C_{1,\gamma} = [0, \frac{1-\gamma}{2}] \cup [\frac{1+\gamma}{2}, 1]$. We then define $C_\gamma := \bigcap_{n=1}^{\infty} C_{n,\gamma}$.

(a) *(3 points).*
Prove that C_γ is Lebesgue measurable with measure $m(C_\gamma) = 1 - \dfrac{\gamma}{1-2\gamma}$.

(b) *(2 points).*
Now fix $\gamma = \frac{1}{3}$ and consider $C := C_{\frac{1}{3}}$, which corresponds to the Cantor set. Prove or disprove that there exists a sequence of numbers $\{x_i\}_{i\in\mathbb{N}}, \dots \in \mathbb{R}$ such that $[0,1] \subset \bigcup_{i=1}^{\infty} (C + x_i)$.

Question 7: *This question is worth 9 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9

(a) *(2 points).*
Prove the dominated convergence theorem in the following setting: let $f_1, f_2, \dots$ be a sequence of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ which converge pointwise. Suppose also that there is an absolutely integrable function $F : \mathbb{R} \to [0, \infty]$ such that $|f_n(x)| \leq F(x)$ for every $x \in \mathbb{R}$ and all $n = 1, 2, 3, \dots$. Then

$$\int_{\mathbb{R}} \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_{\mathbb{R}} f_n.$$

(b) *(2 points).*
For $n \in \mathbb{N} = \{1, 2, \dots\}$, let

$$f_n(x) := \left(\frac{1}{n^2} - \frac{e^{-x}}{n^3}\right) \mathbb{1}_{[1,n]}(x), \qquad \text{for all } x \geq 1,$$

where $\mathbb{1}_{[1,n]}$ denotes the characteristic function of the interval $[1, n]$.
Compute $\lim_{n\to+\infty} f_n(x)$ for all $x \in \mathbb{R}$ and prove that $|f_n(x)| \leq \dfrac{1}{x^2}$ for all $x \in [1, +\infty)$ and for all $n \in \mathbb{N}$.

(c) *(1 point).*
Compute $\displaystyle\lim_{n\to+\infty} \int_{\mathbb{R}} f_n(x)\,\mathrm{d}x$.

(d) *(3 points).*
Prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{f_n(x)}{n^\alpha} \in L^1([1, +\infty)) \iff \alpha > 0.$

(e) *(1 point).*
Verify if the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the hypothesis of the monotone convergence theorem.

Question 8: *This question is worth 3 points.*

☐0 ☐1 ☐2 ☐3

Suppose that $f : [0,1] \longrightarrow (0,+\infty)$ is a positive measurable function and fix $\theta \in (0,1]$. Show that

$$\inf \left\{ \int_E f(x)\,\mathrm{d}x : \ E \subseteq [0,1],\ E \text{ is measurable with } m(E) \geqslant \theta \right\} > 0.$$