Teacher : Maria Colombo
Analysis 4 (MATH-205)
26/04/2023
Duration : 90 minutes

100

XXX-5

SCIPER : **FAKE-5**

Do not turn the page before the start of the exam. This document is double-sided, has 8 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A5 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  +1 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Let $f(x, y, z) = xy$ and $A = \{(x, y, z) : x \in (0, 1), y \in (0, 1), 0 \leq z \leq 2 - x - y\}$.
What is the value of $\int_A f(x, y, z) dx dy dz$?

- ☐ $1/6$
- ☐ 1
- ☐ $1/4$
- ☐ 0

Question 2 Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable functions and V be the Vitali set. Which of the following statements is **true**?

- ☐ $\mathbb{1}_V$ is measurable
- ☐ $f \circ g$ is measurable
- ☐ Every bounded function is measurable
- ☐ Every increasing function is measurable

Question 3 Let $f \in C_c^0(\mathbb{R})$ and define $f_n(x) = nf(nx)$ for any $n \in \mathbb{N}$.
Which of the following statements is **true**?

- ☐ $\lim_{n \to +\infty} \|f_n\|_{L^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})}$
- ☐ $\lim_{n \to +\infty} \|f_n\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$
- ☐ $\lim_{n \to +\infty} \|f_n - f\|_{L^1(\mathbb{R})} = 0$
- ☐ $\lim_{n \to +\infty} \|f_n\|_{L^2(\mathbb{R})} = 0$

Question 4 Let $I_k \subset \mathbb{R}$ be open intervals for any $k \in \mathbb{N}$ and be such that $\mathbb{Q} \cap [0, 1] \subset \bigcup_{k=0}^{\infty} I_k$.
Which of the following statements is **true**?

- ☐ If there exists N such that $I_k = \emptyset$ for any $k \geq N$, then $\sum_{k=0}^{\infty} m^*(I_k) \geq 1$
- ☐ $\sum_{k=0}^{\infty} m^*(I_k) > 1$
- ☐ If there exists N such that $I_k = \emptyset$ for any $k \geq N$, then $\sum_{k=0}^{\infty} m^*(I_k) < 1$
- ☐ $\sum_{k=0}^{\infty} m^*(I_k) \leq 1$

Question 5 Let $f(x) = 2x - 1$ and $\varphi \in C_c^0(\mathbb{R})$ be a continuous function with compact support and $\int_{\mathbb{R}} \varphi(x) dx = 1$. Let $g(x) = \int_{\mathbb{R}} f(x - y)\varphi(y) dy$.
Which of the following statements is **true**?

- ☐ There exists a constant $c \in \mathbb{R}$ depending on φ such that $g(x) = cx - 1$
- ☐ There exists a constant $c \in \mathbb{R}$ depending on φ such that $g(x) = 2x + c$ for any $x \in \mathbb{R}$
- ☐ $g(x) = 1$ for any $x \in \mathbb{R}$
- ☐ $g(x) = +\infty$ for any $x \in \mathbb{R}$

Second part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 6: *This question is worth 9 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9

Let $p \in [1, \infty)$.

(a) *(2 points).* State the dominated convergence theorem.

(b) *(4 points).* Prove that for any Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^p(0,1)$ there exist a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and $f, F \in L^p(0,1)$ such that $|f_{n_k}(x)| \leq F(x)$ for a.e. $x \in (0,1)$ and $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for a.e. $x \in (0,1)$.

(c) *(3 points)* Prove that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p((0,1))$ and $f \in L^p((0,1))$ such that the following two properties hold:

$$\|f_n - f\|_{L^p} \to 0 \qquad \text{as } n \to \infty$$

and there are no $F \in L^p(0,1)$ such that $f_n(x) \leq F(x)$ for any $n \in \mathbb{N}$ and for a.e. $x \in (0,1)$.

Hint: For (c), you may consider the functions $f_n = n^{\frac{1}{2p}} 1_{I_n}$, and $I_{2^m+i} = [i/2^m, (i+1)/2^m]$ for every $m \in \mathbb{N}$, $i \in 0, \ldots, 2^m - 1$, namely $I_1 = [0,1]$, $I_2 = [0,1/2]$, $I_3 = [1/2,1]$, $I_4 = [0,1/4]$, $I_5 = [1/4,2/4]$...

Question 7: *This question is worth 8 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

Let

$$f_n(x) = \frac{n}{x} \sin\left(\frac{x}{n}\right) e^{-x}, \quad x > 0$$

(a) *(1 point).* For every $x > 0$ prove that $\lim_{n\to\infty} f_n(x) = e^{-x}$.

(b) *(3 points).* Compute $\lim_{n\to\infty} \int_0^\infty f_n(x)dx$.

(c) *(4 points).* Using $\sin(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ rewrite $f_2(x)$ as a series and show that

$$\int_0^\infty f_2(x)dx = \sum_{n=0}^\infty \int_0^\infty \frac{(-1)^n x^{2n}}{4^n[(2n+1)!]} e^{-x} dx = \sum_{n=0}^\infty \frac{(-1)^n}{4^n(2n+1)}.$$

Hint: For point (c). To rigorously justify the first equality it may be useful to recall that $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ for any $x \in \mathbb{R}$.