Serie 9

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. For $c \in \mathbb{R}$, consider the PDEs

$$\partial_{tt}u - c^2 u_{xx} = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2\,. \tag{1}$$

$$\partial_t u - c u_{xx} = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2\,. \tag{2}$$

Decide for each of the following functions which PDE they solve.

a) $u(x,t) = \sin(x - ct)$,

b) $u(x,t) = \log(x + ct)$ for $x + ct > 0$,

c) $u(x,t) = x^2 + 2ct$,

d) $u(x,t) = \cos(ax)\sin(cat)$ for $a \in \mathbb{R}$,

e) $u(x,t) = e^{-ct}\sin(x)$,

f) $u(x,t) = e^{ct}\cosh(x)$,

g) $u(x,t) = e^{-a^2 ct}\cos(ax)$ for $a \in \mathbb{R}$,

h) $u(x,t) = e^{x+ct} + e^{x-ct}$.

Exercise 2. Consider, for some $F \in C(\mathbb{R}^2)$ fixed, the two PDEs

$$u_x - e^{-2t}u_{xt} = F(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^2\,. \tag{3}$$

$$u_x - e^{-2t}u_{xt} = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2\,. \tag{4}$$

Assume that $v = v(x,t)$ solves (3) and $w = w(x,t)$ solves (4). Which of the following statements is/are true?

(i) $v + w$ solves (3).

(ii) $\alpha v + \beta w$ solves (3) for any $\alpha, \beta \in \mathbb{R}$.

(iii) $v_\lambda(x,t) := v(\lambda x, t)$ solves (3) for any $\lambda > 0$.

(iv) $\widetilde{w}(x,t) := w(x,-t)$ solves (4).

Exercise 3. Let $u\colon :\mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Express $\partial_{xy}u$ in polar coordinates.

Hint: The formula we look for is

$$\partial_{xy}u(r\cos\theta, r\sin\theta) = \Bigg[\partial_{rr}w\cos\theta\sin\theta + \frac{1}{r}\partial_{r\theta}w(\cos^2\theta - \sin^2\theta) - \frac{1}{r^2}\partial_{\theta\theta}w\cos\theta\sin\theta - \frac{1}{r}\partial_r w\cos\theta\sin\theta + \frac{1}{r^2}\partial_\theta w(\sin^2\theta - \cos^2\theta)\Bigg](r,\theta).$$

To prove it, set $w(r,\theta) = u(r\cos\theta, r\sin\theta)$ and compute $\partial_{r\theta}w$ in terms of ∇u and $\nabla^2 u$.

Exercise 4. Prove that the space of continuous, 1−periodic functions $C^0(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is dense in $L^2((0,1);\mathbb{C})$.

Exercise 5.

(i) Express the function $v(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ in polar coordinates and compute its radial derivative ∂_r.

(ii) Compute Δv both in standard coordinates and in polar coordinates.

(iii) Compute $\sup_{(x,y)\in B_1\setminus\{0\}} v(x,y)$.

(iv) Compute the unique C^2-solution w of the boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } B_1, \\ w = v & \text{on } \partial B_1. \end{cases} \tag{5}$$

Remark: We expect only a formal derivation of the solution and you don't need to prove uniqueness.

(v) Write w both in polar and standard coordinates.

Hints:

- For (ii) show that $\Delta v = 0$ in $B_1 \setminus \{0\}$.
- For (iv), start by finding all solutions (in polar coordinates) to the equation "$\Delta w = 0$ in B_1" which are separable; that is make the ansatz $w(r,\theta) = \varphi(r)\psi(\theta)$. Finally, find the unique solution by looking at the boundary condition.