Serie 9

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. For $c \in \mathbb{R}$, consider the PDEs

$$\partial_{tt}u - c^2 u_{xx} = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2 . \qquad (1)$$

$$\partial_t u - c u_{xx} = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2 . \qquad (2)$$

Decide for each of the following functions which PDE they solve.

a) $u(x,t) = \sin(x - ct)$,

b) $u(x,t) = \log(x + ct)$ for $x + ct > 0$,

c) $u(x,t) = x^2 + 2ct$,

d) $u(x,t) = \cos(ax)\sin(cat)$ for $a \in \mathbb{R}$,

e) $u(x,t) = e^{-ct}\sin(x)$,

f) $u(x,t) = e^{ct}\cosh(x)$,

g) $u(x,t) = e^{-a^2ct}\cos(ax)$ for $a \in \mathbb{R}$,

h) $u(x,t) = e^{x+ct} + e^{x-ct}$.

Solution:

a) (1)

b) (1)

c) (2)

d) (1)

e) (2)

f) (2)

g) (2)

h) (1)

Exercise 2. Consider, for some $F \in C(\mathbb{R}^2)$ fixed, the two PDEs

$$u_x - e^{-2t}u_{xt} = F(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^2 . \qquad (3)$$

$$u_x - e^{-2t}u_{xt} = 0 \qquad \text{for } (x,t) \in \mathbb{R}^2 . \qquad (4)$$

Assume that $v = v(x,t)$ solves (3) and $w = w(x,t)$ solves (4). Which of the following statements is/are true?

(i) $v + w$ solves (3).

(ii) $\alpha v + \beta w$ solves (3) for any $\alpha, \beta \in \mathbb{R}$.

(iii) $v_\lambda(x,t) := v(\lambda x, t)$ solves (3) for any $\lambda > 0$.

(iv) $\widetilde{w}(x,t) := w(x,-t)$ solves (4).

Solution:

(i) True.

(ii) False.

(iii) False.

(iv) False.

Exercise 3. Let $u\colon :\mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Express $\partial_{xy} u$ in polar coordinates.

Hint: The formula we look for is

$$\partial_{xy} u(r\cos\theta, r\sin\theta) = \Bigg[\partial_{rr} w \cos\theta \sin\theta + \frac{1}{r}\partial_{r\theta} w(\cos^2\theta - \sin^2\theta) - \frac{1}{r^2}\partial_{\theta\theta} w \cos\theta\sin\theta \\ - \frac{1}{r}\partial_r w \cos\theta\sin\theta + \frac{1}{r^2}\partial_\theta w(\sin^2\theta - \cos^2\theta)\Bigg](r,\theta).$$

To prove it, set $w(r,\theta) = u(r\cos\theta, r\sin\theta)$ and compute $\partial_{r\theta} w$ in terms of ∇u and $\nabla^2 u$.

Solution: Recall from the lecture that

$$\begin{aligned}
\partial_r w(r,\theta) &= \partial_x u(r\cos\theta, r\sin\theta)\cos\theta + \partial_y u(r\cos\theta, r\sin\theta)\sin\theta, \\
\partial_\theta w(r,\theta) &= -\partial_x u(r\cos\theta, r\sin\theta) r\sin\theta + \partial_y u(r\cos\theta, r\sin\theta) r\cos\theta, \\
\partial_{rr} w(r,\theta) &= \partial_{xx} u(r\cos\theta, r\sin\theta)\cos^2\theta + 2\partial_{xy} u(r\cos\theta, r\sin\theta)\cos\theta\sin\theta \\
&\quad + \partial_{yy} u(r\cos\theta, r\sin\theta)\sin^2\theta,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{r^2}\partial_{\theta\theta} w(r,\theta) &= \partial_{xx} u(r\cos\theta, r\sin\theta)\sin^2\theta - 2\partial_{xy} u(r\cos\theta, r\sin\theta)\cos\theta\sin\theta \\
&\quad + \partial_{yy} u(r\cos\theta, r\sin\theta)\cos^2\theta - \frac{1}{r}\partial_r w(r,\theta).
\end{aligned}$$

Now, we compute the derivative $\partial_{r\theta}$:

$$\begin{aligned}
\partial_r[\partial_\theta w(r,\theta)] &= \partial_r[-\partial_x u(r\cos\theta, r\sin\theta) r\sin\theta + \partial_y u(r\cos\theta, r\sin\theta) r\cos\theta] \\
&= [-\partial_{xx} u(r\cos\theta, r\sin\theta) + \partial_{yy} u(r\cos\theta, r\sin\theta)] r\cos\theta\sin\theta \\
&\quad + \partial_{xy} u(r\cos\theta, r\sin\theta) r(\cos^2\theta - \sin^2\theta) \\
&\quad - \partial_x u(r\cos\theta, r\sin\theta)\sin\theta + \partial_y u(r\cos\theta, r\sin\theta)\cos\theta
\end{aligned}$$

so that

$$
\begin{aligned}
\frac{1}{r}\partial_{r\theta}w(r,\theta) &= [-\partial_{xx}u(r\cos\theta, r\sin\theta) + \partial_{yy}u(r\cos\theta, r\sin\theta)]\cos\theta\sin\theta \\
&\quad + \partial_{xy}u(r\cos\theta, r\sin\theta)(\cos^2\theta - \sin^2\theta) \\
&\quad + \frac{1}{r^2}\partial_\theta w(r,\theta).
\end{aligned}
$$

Thus, (to lighten the notation we write w without its arguments)

$$
\begin{aligned}
&\partial_{rr}w\cos\theta\sin\theta + \frac{1}{r}\partial_{r\theta}w(\cos^2\theta - \sin^2\theta) - \frac{1}{r^2}\partial_{\theta\theta}w\cos\theta\sin\theta \\
&\quad - \frac{1}{r}\partial_r w\cos\theta\sin\theta + \frac{1}{r^2}\partial_\theta w(\sin^2\theta - \cos^2\theta) \\
&= [\partial_{xx}u\cos^2\theta + 2\partial_{xy}u\cos\theta\sin\theta + \partial_{yy}u\sin^2\theta]\cos\theta\sin\theta \\
&\quad + [[-\partial_{xx}u + \partial_{yy}u]\cos\theta\sin\theta + \partial_{xy}u(\cos^2\theta - \sin^2\theta) + \frac{1}{r^2}\partial_\theta w](\cos^2\theta - \sin^2\theta) \\
&\quad - [\partial_{xx}u\sin^2\theta - 2\partial_{xy}u\cos\theta\sin\theta + \partial_{yy}u\cos^2\theta - \frac{1}{r}\partial_r w]\cos\theta\sin\theta \\
&\quad - \frac{1}{r}\partial_r w\cos\theta\sin\theta + \frac{1}{r^2}\partial_\theta w(\sin^2\theta - \cos^2\theta) \\
&= \partial_{xx}u\underbrace{[\cos^3\theta\sin\theta - (\cos^2\theta - \sin^2\theta)\cos\theta\sin\theta - \sin^3\theta\cos\theta]}_{=0} \\
&\quad + \partial_{xy}u[4\cos^2\theta\sin^2\theta + (\cos^2\theta - \sin^2\theta)^2] \\
&\quad + \partial_{yy}u\underbrace{[\sin^3\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\cos\theta\sin\theta - \cos^3\theta\sin\theta]}_{=0} \\
&= \partial_{xy}u
\end{aligned}
$$

where the last equality follows from $4\cos^2\theta\sin^2\theta + (\cos^2\theta - \sin^2\theta)^2 = (\cos^2\theta + \sin^2\theta)^2 = 1$.

Exercise 4. Prove that the space of continuous, $1-$periodic functions $C^0(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is dense in $L^2((0,1);\mathbb{C})$.

Solution: We extend $f \in L^2((0,1))$ to $f \in L^2(\mathbb{R})$ by imposing $f = 0$ outside $(0,1)$. We know from the theory that we can approximate a function in $L^2(\mathbb{R})$ with

$$\{f_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}) \text{ such that } f_n \to f \text{ in } L^2(\mathbb{R}).$$

Our $\{f_n\}_{n\in\mathbb{N}}$ have support in $\mathbb{R}$, and we need to generate from them continuous functions with compact support in $(0,1)$. To do so, we can define a continuous cutoff function

$$\phi(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x \leq 1/2 \\ 2x-1, & x \in (1/2, 1) \end{cases}$$

and localize the support in $(0,1)$ by multiplying f_n by the cutoff g_n, defined as

$$g_n(x) \coloneqq \phi(n\text{dist}(x,(0,1)^C)),$$

where

$$\text{dist}(x,(0,1)^C) = \begin{cases} 0, & x \geq 1, x \leq 0 \\ x, & x \in (0,1/2] \\ 1-x, & x \in (1/2,1). \end{cases}$$

Thanks to the definition of the cutoff, the functions $f_n g_n$ are continuous, have compact support in $(0,1)$ and we can prove that

$$f_n g_n \to f \quad \text{in } L^2(0,1).$$

Indeed,

$$\begin{aligned} ||f_n g_n - f||_{L^2} &= ||f_n g_n - f g_n + f g_n - f||_{L^2} \\ &\leq ||f(g_n - 1)||_{L^2} + ||(f_n - f)g_n||_{L^2} \\ &\leq ||f(g_n - 1)||_{L^2} + ||f_n - f||_{L^2} \underbrace{||g_n||_{L^\infty}}_{\leq 1}. \end{aligned}$$

Passing to the limit for $n \to +\infty$, we can apply the dominated convergence theorem on the first term, because $|g_n - 1| \leq 2$, and we have by hypothesis that $||f_n - f||_{L^2} \to 0$.

Hence, we conclude the proof and get the approximation with the family $\{f_n g_n\}_{n\in\mathbb{N}}$, that is continuous and compactly supported in $(0,1)$.

Exercise 5.

(i) Express the function $v(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ in polar coordinates and compute its radial derivative ∂_r.

(ii) Compute Δv both in standard coordinates and in polar coordinates.

(iii) Compute $\sup_{(x,y)\in B_1\setminus\{0\}} v(x,y)$.

(iv) Compute the unique C^2-solution w of the boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } B_1, \\ w = v & \text{on } \partial B_1. \end{cases} \tag{5}$$

Remark: We expect only a formal derivation of the solution and you don't need to prove uniqueness.

(v) Write w both in polar and standard coordinates.

Hints:

- For (ii) show that $\Delta v = 0$ in $B_1 \setminus \{0\}$.

- For (iv), start by finding all solutions (in polar coordinates) to the equation "$\Delta w = 0$ in B_1" which are separable; that is make the ansatz $w(r, \theta) = \varphi(r)\psi(\theta)$. Finally, find the unique solution by looking at the boundary condition.

Solution:

(i) In polar coordinates,

$$u(r, \theta) = v(r\cos\theta, r\sin\theta) = \frac{\cos^2\theta - \sin^2\theta}{r^2}$$

and

$$\partial_r u(r, \theta) = -\frac{2}{r^3}(\cos^2\theta - \sin^2\theta).$$

Note that the function is not well defined in $(0, 0)$.

(ii) First in standard coordinates: with $(x, y) \neq (0, 0)$

$$\partial_x v(x, y) = \frac{2x(x^2 + y^2) - 4x(x^2 - y^2)}{(x^2 + y^2)^4} = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3},$$

$$\partial_{xx} v(x, y) = \frac{(-6x^2 + 6y^2)(x^2 + y^2)^3 - 6x(-2x^3 + 6xy^2)(x^2 + y^2)^2}{(x^2 + y^2)^6} = \frac{6x^4 - 36x^2y^2 + 6y^4}{(x^2 + y^2)^4}$$

and

$$\partial_{yy} v(x, y) = -\frac{6y^4 - 36x^2y^2 + 6y^4}{(x^2 + y^2)^4}.$$

Thus,

$$\Delta v(x, y) = \partial_{xx} v(x, y) + \partial_{yy} v(x, y) = 0 \quad \forall (x, y) \in B_1 \setminus \{0\}.$$

And in polar coordinates: for $r > 0$

$$\partial_{rr} u(r, \theta) = \partial_r(-\frac{2}{r^3}(\cos^2\theta - \sin^2\theta)) = \frac{6}{r^4}(\cos^2\theta - \sin^2\theta),$$

$$\partial_\theta u(r, \theta) = -\frac{4}{r^4}\cos\theta\sin\theta$$

and

$$\partial_{\theta\theta} u(r, \theta) = -\frac{4}{r^2}(\cos^2\theta - \sin^2\theta).$$

Thus,

$$\Delta u(r, \theta) = \partial_{rr} u(r, \theta) + \frac{1}{r}\partial_r u(r, \theta) + \frac{1}{r^2}\partial_{\theta\theta} u(r, \theta) = 0 \quad \text{for } r > 0.$$

(iii) When $y = 0$,

$$v(x, 0) = \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Thus letting $x \to 0$, we deduce $\sup_{(x,y) \in B_1 \setminus \{0\}} v(x, y) = \infty$

(iv) In order to find a solution w, we try to find a solution in polar coordinates. We assume at first that the solution is separable, i.e. there are functions φ and ψ such that $w(r, \theta) = \varphi(r)\psi(\theta)$.

The problem (5) becomes

$$\begin{cases} r^2\frac{\varphi''(r)}{\varphi(r)} + r\frac{\varphi'(r)}{\varphi(r)} = -\frac{\psi''(\theta)}{\psi(\theta)} = \lambda; \\ \varphi(1)\psi(\theta) = \cos^2\theta - \sin^2\theta; \\ \varphi(r)\psi(0) = \varphi(r)\psi(2\pi); \\ \varphi(r)\psi'(0) = \varphi(r)\psi'(2\pi), \end{cases} \tag{6}$$

for some suitable λ. This yields two ODE's

$$\begin{cases} \psi''(\theta) + \lambda\psi(\theta) = 0; \\ \psi(0) = \psi(2\pi); \\ \psi'(0) = \psi'(2\pi), \end{cases} \tag{7}$$

and

$$\frac{r^2\varphi''(r) + r\varphi'(r)}{\varphi(r)} = \lambda. \tag{8}$$

The problem (**??**) admits a solution only if $\lambda = n^2$, $n \in \mathbb{N}_{\geq 0}$ and the solution is given by $\psi(\theta) = \alpha_n \cos(n\theta) + \beta_n \sin(n\theta)$. When $\lambda = n^2$ the solution to (**??**) is given by

$$\varphi(r) = \begin{cases} \eta_0 + \delta_0 \log(r) & \text{if } n = 0; \\ \eta_n r^n + \delta_n r^{-n} & \text{if } n \neq 0. \end{cases} \tag{9}$$

Since we are searching for a continuous function, $\delta_n = 0$ for all n, and thus

$$w_n(r,\theta) = [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]\eta_n r^n$$

is a solution to $\Delta w_n = 0$. In order to find a general solution to $\Delta w = 0$, we write the solution as a superposition of w_n i.e.

$$w(r,\theta) = \eta_0\alpha_0 + \sum_{n=1}^{\infty}[\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]\eta_n r^n.$$

We determine the constants α_n, β_n, η_n by looking at the boundary condition

$$w(1,\theta) = \cos^2\theta - \sin^2\theta = \cos(2\theta).$$

From this, we deduce $w(r,\theta) = r^2\cos(2\theta) = r^2(\cos^2\theta - \sin^2\theta)$.

(v) We already know that in polar coordinates $w(r,\theta) = r^2\cos(2\theta)$ and in standard coordinates we get $x^2 - y^2$.