Serie 8

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1 (Orthonormal system)**.** Recall that $L^2((0,T);\mathbb{C})$ can be endowed with the scalar product

$$\langle f, g\rangle := \int_0^T f\bar{g}\,dx.$$

(i) Show that the functions $\{e_n(x) := \mathrm{e}^{2\pi i n x}\}_{n\in\mathbb{Z}}$ form an orthonormal system in $L^2((0,1);\mathbb{C})$, namely

$$\langle e_n, e_m\rangle = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m\,. \end{cases} \tag{1}$$

(ii) Show that the functions $\left\{\sqrt{\frac{2}{T}}\sin(\frac{2\pi n x}{T}), \sqrt{\frac{2}{T}}\cos(\frac{2\pi n x}{T})\right\}_{n\in\mathbb{N}}$ form an orthonormal system of $L^2((0,T);\mathbb{R})$.

Exercise 2. For $f \in L^1(\mathbb{R}^n)$, we define for $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx\,. \tag{2}$$

We will call $\hat{f}$ the Fourier transform of f. Prove that

(i) $\hat{f}$ is well-defined, i.e. that the integral on the right-hand side of (2) converges for every $\xi \in \mathbb{R}^n$,

(ii) $\hat{f}$ is a bounded function on $\mathbb{R}^n$,

(iii) $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

Hint: Write (and justify it) that for any $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} \{f(x) - f(x-\xi')\}\, e^{-2\pi i x\cdot\xi}\,dx \text{ with } \xi' = \frac{\xi}{2|\xi|^2},$$

and use a previous exercise.

Exercise 3. Show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that

$$f * I = f \text{ for all } f \in L^1(\mathbb{R}^n).$$

Hint: Use the Fourier transform and Exercise 2 of Series 8.

Exercise 4. We introduce the Fejer kernel $F_N \colon \mathbb{R} \to \mathbb{C}$ defined by

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \mathrm{e}^{2\pi i n x}.$$

The Fejer kernel will play a central role in the proof of the uniform convergence of the Fourier series. Before that, we establish some of its important properties.

(i) Show that F_N has no imaginary part for every x and plot with Wolfram-alpha F_1, F_2, F_3, F_4.

(ii) Show that

$$F_N(x) = \frac{1}{N} \left| \sum_{n=0}^{N-1} e^{2\pi i n x} \right|^2 = \frac{1}{N} \left(\frac{\sin(\pi N x)}{\sin(\pi x)} \right)^2.$$

(iii) Show that $F_N \geq 0$, that $\int_0^1 F_N \, dx = 1$ and that for every $\varepsilon, \delta > 0$ there exists N_0 such that $\forall N \geq N_0$ we have

$$F_N(x) < \varepsilon \quad \forall x \in (\delta, 1 - \delta).$$

(iv) If $f, g \colon \mathbb{R} \to \mathbb{R}$ are continuous and 1-periodic, we define the periodic convolution by

$$(f * g)(x) := \int_0^1 f(y) g(x - y) \, dy,$$

for $x \in \mathbb{R}$. Using (iii), show that for f continuous and 1-periodic, we have

$$\|f - f * F_N\|_{L^\infty} \to 0 \text{ as } N \to \infty.$$

(v) For $f : \mathbb{R} \to \mathbb{R}$ continuous and 1-periodic, we define the Nth Cesaro mean by

$$\Phi_N f(x) := \frac{1}{N} \sum_{n=0}^{N-1} S_n f(x),$$

where $S_n f(x) := \sum_{j=-n}^{n} \langle f, e_j \rangle e_j(x)$ (recall from Exercise 1 that $e_j(x) := e^{2\pi i j x}$ and $\langle f, e_j \rangle := \int_0^1 f(x) e^{-2\pi i j x} \, dx$). Using the definition of periodic convolution in (iv), show that

$$\Phi_n f(x) := (F_N * f)(x).$$

Hints:

- For (ii), use the geometric series formula.
- For (iv), observe (and prove) that f is uniformly continuous.

Exercise 5. Let $p \in [1, \infty)$, $f_k, f \in L^p(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$ measurable and denote $p' \in [1, \infty]$ the conjugate exponent, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We say that "f_k converges to f weakly in $L^p(\Omega)$" or in short "$f_k \rightharpoonup f$" if for all $g \in L^{p'}(\Omega)$ we have :

$$\lim_{k \to \infty} \int_\Omega f_k g \, dx = \int_\Omega f g \, dx.$$

Show that the following holds for $p \in [1, \infty)$, $f, g, f_k \in L^p(\Omega)$:

(i) $f_k \to f$ in $L^p(\Omega) \Rightarrow f_k \rightharpoonup f$ in $L^p(\Omega)$, that is strong convergence in $L^p(\Omega)$ implies weak convergence in $L^p(\Omega)$.

(ii) $f_k \rightharpoonup f$ and $f_k \rightharpoonup g$ in $L^p(\Omega) \Rightarrow f = g$ in $L^p(\Omega)$, that is the weak limit is unique.

(iii) $f_k \rightharpoonup f \Rightarrow \|f\|_{L^p(\Omega)} \leq \liminf_{k\to\infty} \|f_k\|_{L^p(\Omega)}$ that is the $L^p(\Omega)$-norm is lower semi-continuous with respect to weak convergence.

(iv) If $f \in L^p(\Omega)$, we have that

$$\|f\|_{L^p(\Omega)} = \sup\left\{ \int_\Omega f g \, dx : g \in L^{p'}(\Omega), \|g\|_{L^{p'}(\Omega)} \leq 1 \right\}. \tag{6}$$

(v) ($\star$) If $f : \Omega \to \mathbb{R}$ is measurable but $f \notin L^p(\Omega)$ we set $\|f\|_{L^p(\Omega)} = +\infty$. Prove that (6) holds even in this case.

Hints:

- For (iii), use and prove the following "convexity" inequality:

$$|z|^p \geq |y|^p + y^\star(y)(z - y) \quad \forall z, y \in \mathbb{R} \tag{7}$$

  where

$$y^\star(y) = \begin{cases} py|y|^{p-2} & \text{if } y \neq 0; \\ 0 & \text{if } y \neq 0. \end{cases}$$

- For (v), construct a sequence of test functions $g_k \in L^{p'}(\Omega)$ such that

$$\left| \int_\Omega f g_k \, dx \right| \geq k.$$