Serie 8
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1 (Orthonormal system)**.** Recall that $L^2((0,T);\mathbb{C})$ can be endowed with the scalar product

$$\langle f, g\rangle := \int_0^T f\bar{g}\,dx.$$

(i) Show that the functions $\{e_n(x) := \mathrm{e}^{2\pi inx}\}_{n\in\mathbb{Z}}$ form an orthonormal system in $L^2((0,1);\mathbb{C})$, namely

$$\langle e_n, e_m\rangle = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m\,. \end{cases} \tag{1}$$

(ii) Show that the functions $\left\{\sqrt{\frac{2}{T}}\sin(\frac{2\pi nx}{T}), \sqrt{\frac{2}{T}}\cos(\frac{2\pi nx}{T})\right\}_{n\in\mathbb{N}}$ form an orthonormal system of $L^2((0,T);\mathbb{R})$.

Solution:

(i) Indeed, using that $\overline{e^{2\pi imx}} = e^{-2\pi imx}$, we have

$$\langle e_n, e_m\rangle = \int_0^1 \mathrm{e}^{2\pi inx}\overline{\mathrm{e}^{2\pi imx}}\,dx = \int_0^1 \mathrm{e}^{2\pi i(n-m)x}\,dx = \begin{cases} \int_0^1 dx = 1 & \text{if } n = m, \\ \int_0^1 \mathrm{e}^{2\pi i(n-m)x}\,dx = 0 & \text{if } n \neq m. \end{cases}$$

(ii) Set $a_n(x) := \sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n}{T}x\right)$ and $b_n(x) := \sqrt{\frac{2}{T}}\sin\left(\frac{2\pi n}{T}x\right)$. We need to show that $\langle a_n, b_m\rangle = 0$, $\langle a_n, a_m\rangle = \delta_{nm}$ and $\langle b_n, b_m\rangle = \delta_{nm}$ for all $n, m \in \mathbb{N}$. Recall the following trigonometric identities:

$$2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y),$$
$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y),$$
$$2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y).$$

We first prove $\langle a_n, b_m \rangle = 0$ for all $n, m \in \mathbb{N}$: we compute using the third identity

$$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)\,dx = \frac{1}{2}\int_0^T \left[\sin\left(\frac{2\pi}{T}(n-m)x\right) + \sin\left(\frac{2\pi}{T}(n+m)x\right)\right]\,dx$$
$$= \frac{T}{4\pi}\int_0^{2\pi} \sin\left((n-m)x\right)\,dx + \frac{T}{4\pi}\int_0^{2\pi} \sin\left((n+m)x\right)\,dx.$$

For any $m, n \in \mathbb{N}$, both integrals take the value 0 and we conclude. The remaining equations are proved in a similar way using the other two trigonometric identities.

Exercise 2. For $f \in L^1(\mathbb{R}^n)$, we define for $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx\,. \tag{2}$$

We will call $\hat{f}$ the Fourier transform of f. Prove that

(i) $\hat{f}$ is well-defined, i.e. that the integral on the right-hand side of (2) converges for every $\xi \in \mathbb{R}^n$,

(ii) $\hat{f}$ is a bounded function on $\mathbb{R}^n$,

(iii) $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

Hint: Write (and justify it) that for any $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} \{f(x) - f(x-\xi')\}\, e^{-2\pi i x\cdot\xi}\,dx \text{ with } \xi' = \frac{\xi}{2|\xi|^2},$$

and use a previous exercise.

Solution: As for the well-definition (i), we use the monotonicity of the integral together with the fact $|e^{-2\pi i x\cdot\xi}| = 1$ for every $x, \xi \in \mathbb{R}^n$ to estimate

$$|\hat{f}(\xi)| = \left|\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx\right| \leq ||f||_{L^1(\mathbb{R}^n)}$$

for every $\xi \in \mathbb{R}^n$. Hence the right-hand side of (2) converges for every $\xi \in \mathbb{R}^n$ and $\hat{f}$ is well-defined. The very same computation shows that

$$\sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \leq ||f||_{L^1(\mathbb{R}^n)}\,,$$

i.e. $\hat{f}$ is a bounded function of ξ. Regarding (iii), we note that

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$$
$$= \int_{\mathbb{R}^n} f(z-\xi')e^{-2\pi i (z-\xi')\cdot\xi}\,dz$$
$$= -\int_{\mathbb{R}^n} f(z-\xi')e^{-2\pi i z\cdot\xi}\,dz \text{ with } \xi' = \frac{\xi}{2|\xi|^2},$$

hence

$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} \{f(x) - f(x-\xi')\}\, e^{-2\pi i x\cdot\xi}\, dx.$$

Then by monotonicity of the integral

$$|\hat{f}(\xi)| \le \frac{1}{2}\int_{\mathbb{R}^n} |f(x) - f(x-\xi')||e^{-2\pi i x\cdot\xi}|\, dx = \frac{1}{2}\int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\, dx.$$

Finally, since $|\xi'| \to 0$, as $|\xi| \to \infty$, $\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$ due to Exercise 3 of Series 6.

Exercise 3. Show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that

$$f * I = f \text{ for all } f \in L^1(\mathbb{R}^n).$$

Hint: Use the Fourier transform and Exercise 2 of Series 8.

Solution: Suppose for a contradiction that such an I exists. Recall that that for $f, g \in L^1(\mathbb{R}^n)$, we have $\widehat{(f*g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. Thus $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$, and therefore $\hat{I}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$. However, since $I \in L^1(\mathbb{R}^n)$, we have $\hat{I}(\xi) \to 0$ as $|\xi| \to \infty$ (see Exercise 2 of Series 8), giving the desired contradiction.

Exercise 4. We introduce the Fejer kernel $F_N \colon \mathbb{R} \to \mathbb{C}$ defined by

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \mathrm{e}^{2\pi i n x}\,.$$

The Fejer kernel will play a central role in the proof of the uniform convergence of the Fourier series. Before that, we establish some of its important properties.

(i) Show that F_N has no imaginary part for every x and plot with Wolfram-alpha F_1, F_2, F_3, F_4.

(ii) Show that

$$F_N(x) = \frac{1}{N}\left|\sum_{n=0}^{N-1} e^{2\pi i n x}\right|^2 = \frac{1}{N}\left(\frac{\sin(\pi N x)}{\sin(\pi x)}\right)^2.$$

(iii) Show that $F_N \ge 0$, that $\int_0^1 F_N\, dx = 1$ and that for every $\varepsilon, \delta > 0$ there exists N_0 such that $\forall N \ge N_0$ we have

$$F_N(x) < \varepsilon \quad \forall x \in (\delta, 1-\delta).$$

(iv) If $f, g \colon \mathbb{R} \to \mathbb{R}$ are continuous and 1-periodic, we define the periodic convolution by

$$(f * g)(x) := \int_0^1 f(y) g(x-y)\, dy,$$

for $x \in \mathbb{R}$. Using (iii), show that for f continuous and 1-periodic, we have

$$\|f - f * F_N\|_{L^\infty} \to 0 \text{ as } N \to \infty.$$

(v) For $f : \mathbb{R} \to \mathbb{R}$ continuous and 1-periodic, we define the Nth Cesaro mean by

$$\Phi_N f(x) := \frac{1}{N} \sum_{n=0}^{N-1} S_n f(x)\,,$$

where $S_n f(x) := \sum_{j=-n}^{n} \langle f, e_j \rangle e_j(x)$ (recall from Exercise 1 that $e_j(x) := e^{2\pi i j x}$ and $\langle f, e_j \rangle := \int_0^1 f(x) e^{-2\pi i j x}\, dx$). Using the definition of periodic convolution in (iv), show that

$$\Phi_n f(x) := (F_N * f)(x)\,.$$

Hints:

- For (ii), use the geometric series formula.
- For (iv), observe (and prove) that f is uniformly continuous.

Solution:

(i) We have

$$\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \mathrm{e}^{2\pi i n x} = 1 + \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) (\mathrm{e}^{2\pi i n x} + \mathrm{e}^{-2\pi i n x}) = 1 + 2 \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) \cos(2\pi n x) \tag{3}$$

which proves that F_N has no imaginary part.

(ii) We begin by proving

$$F_N(x) = \frac{1}{N} \left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi i n x} \right|^2.$$

Indeed,

$$\left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi i n x} \right|^2 = \Big(\sum_{n=0}^{N-1} \mathrm{e}^{2\pi i n x} \Big) \Big(\sum_{n=0}^{N-1} \mathrm{e}^{-2\pi i n x} \Big)$$

so that we easily see that the coefficient in front of each $\mathrm{e}^{2\pi i n x}$ is given by $N - |n|$. Thus,

$$F_N(x) = \frac{1}{N} \left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi i n x} \right|^2. \tag{4}$$

Before continuing with the other equality, notice that $\sum_{n=0}^{m} a^n = \frac{a^{m+1}-1}{a-1}$. Therefore,

$$\sum_{n=0}^{N-1} \mathrm{e}^{2\pi i n x} = \frac{\mathrm{e}^{2\pi i N x} - 1}{\mathrm{e}^{2\pi i x} - 1} = \frac{\mathrm{e}^{\pi i N x}(\mathrm{e}^{i\pi N x} - \mathrm{e}^{-i\pi N x})}{\mathrm{e}^{i\pi x}(\mathrm{e}^{i\pi x} - \mathrm{e}^{-i\pi x})} = \mathrm{e}^{i\pi(N-1)x} \frac{\sin(\pi N x)}{\sin(\pi x)}.$$

Thus, due to (4),

$$F_N(x) = \frac{1}{N}\left(\frac{\sin(\pi N x)}{\sin(\pi x)}\right)^2. \tag{5}$$

(iii) It follows from (5) that $F_N \geq 0$. Moreover, using (3) we have

$$\int_0^1 F_N(x)\,dx = 1 + 2\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)\int_0^1 \cos(2\pi n x)\,dx = 1.$$

Fix now $\varepsilon, \delta > 0$. Since $|\sin(\pi N x)| \leq 1$, we deduce from (5) that whenever $\delta < x < 1-\delta$

$$F_N(x) = \frac{1}{N}\left|\frac{1}{\sin(\pi x)}\right|^2 \leq \frac{1}{N\sin^2(\pi\delta)}.$$

Thus, choosing N large enough, we can make $F_N(x) \leq \varepsilon$ for all $\delta < x < 1-\delta$.

(iv) By periodicity, it is enough to show the convergence for $x \in [0,1]$. Let thus $\varepsilon > 0$ fixed. Since f is bounded, there is M such that $|f(x)| \leq M$ for all $x \in [0,1]$. As a continuous function on a compact set, f is uniformly continuous on $[0,1]$; thus there exists $\delta = \delta(\varepsilon) > 0$ such that for all $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. From (iii), we know that there exists $N_0 = N_0(\varepsilon, \delta)$ such that for all $N \geq N_0$, $F_N(x) \leq \varepsilon$ for all $x \in (\delta, 1-\delta)$. Moreover since $\int_0^1 F_N(x)\,dx = 1$, we have

$$f(x) = \int_0^1 f(x) F_N(y)\,dy.$$

We estimate for $x \in [0,1]$ using the monotonicity of the integral and the positivity of F_N

$$\begin{aligned}|f(x) - f * F_N(x)| &= \left|\int_0^1 f(x)F_N(y)\,dy - \int_0^1 f(x-y)F_N(y)\,dy\right| = \left|\int_0^1 [f(x)-f(x-y)]F_N(y)\,dy\right| \\ &\leq \int_0^1 |f(x)-f(x-y)|F_N(y)\,dy.\end{aligned}$$

We can now split this integral in three parts: for $y \in [0,\delta] \cup [1-\delta, 1]$ we have by choice of δ $|f(x) - f(x-y)| \leq \varepsilon$ where as for $y \in [\delta, 1-\delta]$ we just have the bound $|f(x) - f(x-y)| \leq 2M$. Thus for $N \geq N_0$

$$\begin{aligned}|f(x) - f * F_N(x)| &\leq \int_0^\delta \varepsilon F_N(y)\,dy + \int_\delta^{1-\delta} 2M F_N(y)\,dy + \int_{1-\delta}^1 \varepsilon F_N(y)\,dy \\ &\leq \varepsilon + 2M\varepsilon + \varepsilon = (2M+2)\varepsilon,\end{aligned}$$

where we used $N \geq N_0$ and the choice of N_0 in the second inequality. Thus, for all $N \geq N_0$

$$\|f - f * F_N\|_{L^\infty([0,1])} \leq (2M+2)\varepsilon.$$

We conclude by the arbitrariness of ε.

(v) We compute for $x \in [0,1]$ using the linearity of the integral and the definition of the L^2

scalar product

$$\begin{aligned} f * F_N(x) &= \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N}\Big) \int_0^1 f(y) e^{2\pi i n(x-y)}\, dx \\ &= \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N}\Big) \langle f, e_n \rangle e_n(x) \\ &= \frac{1}{N} \sum_{n=-N}^{N} (N - |n|) \langle f, e_n \rangle e_n(x) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j=-n}^{n} \langle f, e_j \rangle e_j(x) = \Phi_N f(x)\,. \end{aligned}$$

Exercise 5. Let $p \in [1, \infty)$, $f_k, f \in L^p(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$ measurable and denote $p' \in [1, \infty]$ the conjugate exponent, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We say that "f_k converges to f weakly in $L^p(\Omega)$" or in short "$f_k \rightharpoonup f$" if for all $g \in L^{p'}(\Omega)$ we have :

$$\lim_{k\to\infty} \int_\Omega f_k g\, dx = \int_\Omega f g\, dx.$$

Show that the following holds for $p \in [1, \infty)$, $f, g, f_k \in L^p(\Omega)$:

(i) $f_k \to f$ in $L^p(\Omega) \Rightarrow f_k \rightharpoonup f$ in $L^p(\Omega)$, that is strong convergence in $L^p(\Omega)$ implies weak convergence in $L^p(\Omega)$.

(ii) $f_k \rightharpoonup f$ and $f_k \rightharpoonup g$ in $L^p(\Omega) \Rightarrow f = g$ in $L^p(\Omega)$, that is the weak limit is unique.

(iii) $f_k \rightharpoonup f \Rightarrow \|f\|_{L^p(\Omega)} \leq \liminf_{k\to\infty} \|f_k\|_{L^p(\Omega)}$ that is the $L^p(\Omega)$-norm is lower semi-continuous with respect to weak convergence.

(iv) If $f \in L^p(\Omega)$, we have that

$$\|f\|_{L^p(\Omega)} = \sup\left\{ \int_\Omega f g\, dx : g \in L^{p'}(\Omega), \|g\|_{L^{p'}(\Omega)} \leq 1 \right\}. \tag{6}$$

(v) **($\star$)** If $f : \Omega \to \mathbb{R}$ is measurable but $f \notin L^p(\Omega)$ we set $\|f\|_{L^p(\Omega)} = +\infty$. Prove that (6) holds even in this case.

Hints:

- For (iii), use and prove the following "convexity" inequality:

$$|z|^p \geq |y|^p + y^\star(y)(z - y) \quad \forall z, y \in \mathbb{R} \tag{7}$$

where

$$y^\star(y) = \begin{cases} py|y|^{p-2} & \text{if } y \neq 0; \\ 0 & \text{if } y \neq 0. \end{cases}$$

- For (v), construct a sequence of test functions $g_k \in L^{p'}(\Omega)$ such that

$$\left| \int_\Omega f g_k \, dx \right| \geq k.$$

Solution:

(i) If $f_k \to f$ in $L^p(\Omega)$ have $\lim_{k\to\infty} \|f_k - f\|_{L^p(\Omega)} = 0$ so that by Hölder's inequality

$$\lim_{k\to\infty} \left| \int_\Omega f_k g \, dx - \int_\Omega f g \, dx \right| = \lim_{k\to\infty} \left| \int_\Omega (f_k - f) g \, dx \right| \leq \lim_{k\to\infty} \|f_k - f\|_{L^p} \|g\|_{L^{p'}} = 0$$

for all $g \in L^{p'}(\Omega)$. Hence for every test function $g \in L^{p'}(\Omega)$, we have

$$\lim_{k\to\infty} \int_\Omega f_k g \, dx = \int_\Omega f g \, dx,$$

so that $f_k \rightharpoonup f$ in $L^p(\Omega)$.

(ii) We know that

$$\int_\Omega f h \, dx = \lim_{k\to\infty} \int_\Omega f_k h \, dx = \int_\Omega g h \, dx$$

for any $h \in L^{p'}(\Omega)$. Thus

$$\int_\Omega (f - g) h \, dx = 0 \quad \forall h \in L^{p'}(\Omega). \tag{8}$$

We take h to be $h = |f - g|^{p-1} \operatorname{sgn}(f - g)$. Before plugging it into (8), however, we show that $h \in L^{p'}(\Omega)$ and hence is a valid competitor. Indeed, h is measurable and

$$\int_\Omega |h|^{p'} \, dx = \int_\Omega |f - g|^{p'(p-1)} \, dx = \int_\Omega |f - g|^p \, dx < \infty.$$

By plugging this h into (8) we get

$$\|f - g\|_{L^p}^p = \int_\Omega (f - g) h \, dx = 0,$$

which implies $f = g$ in $L^p(\Omega)$.

(iii) We begin by proving the following inequality:

$$|z|^p \geq |y|^p + y^\star(y)(z - y) \quad \forall z, y \in \mathbb{R} \tag{9}$$

where

$$y^\star(y) = \begin{cases} py|y|^{p-2} & \text{if } y \neq 0; \\ 0 & \text{if } y \neq 0. \end{cases} \tag{10}$$

If $z = 0$ or $y = 0$, (9) is trivial. Otherwise, it follows from convexity of $\zeta \mapsto |\zeta|^p$. (Indeed, recall that for f convex and C^1, we have proven that $f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$.) Now let $x \in \Omega$ and put $y = f(x)$, $z = f_k(x)$. By (9) we get

$$|f_k(x)|^p \geq |f(x)|^p + y^\star(f(x))(f_k(x) - f(x)).$$

Integrating over Ω we get

$$\int_\Omega |f_k(x)|^p \, dx \geq \int_\Omega |f(x)|^p \, dx + \int_\Omega y^\star(f(x))(f_k(x) - f(x)) \, dx. \tag{11}$$

Note that $y^\star(f) \in L^{p'}(\Omega)$ so that since $f_k \rightharpoonup f$ the integral on the right converges to 0 as $k \to \infty$. Therefore taking the liminf of (11), we get

$$\liminf_{k\to\infty} \int_\Omega |f_k|^p \, dx \geq \int_\Omega |f|^p \, dx$$

which gives the desired result.

(iv) Let $f \in L^p(\Omega)$. It is clear from Hölder's inequality that

$$\sup \left\{ \int_\Omega fg \, dx : g \in L^{p'}(\Omega), \|g\|_{L^{p'}(\Omega)} \leq 1 \right\} \leq \|f\|_{L^p(\Omega)}.$$

Now we prove the reverse inequality. Actually we prove that there is $g \in L^{p'}(\Omega)$ such that

$$\int_\Omega fg \, dx = \|f\|_{L^p(\Omega)}.$$

If $p = 1$, we define $g\colon \Omega \to \mathbb{R}$ by $g(x) = \operatorname{sgn}(f(x))$. It is clear that $g \in L^\infty(\Omega)$ and

$$\int_\Omega fg \, dx = \int_\Omega |f| \, dx = \|f\|_{L^1(\Omega)}.$$

If $p > 1$, we define $g\colon \Omega \to \mathbb{R}$ by

$$g(x) = \frac{|f(x)|^{p-1}}{\|f\|^{p/p'}_{L^p(\Omega)}} \operatorname{sgn}(f(x)).$$

Then g is measurable and

$$\int_\Omega |g|^{p'} \, dx = \int_\Omega \frac{|f|^{(p-1)p'}}{\|f\|^p_{L^p(\Omega)}} \, dx = \frac{1}{\|f\|_{L^p(\Omega)}} \int_\Omega |f|^p \, dx$$

implying that $g \in L^{p'}(\Omega)$. In addition,

$$\int_\Omega fg \, dx = \int_\Omega \frac{|f|^p}{\|f\|^{p/p'}_{L^p(\Omega)}} \, dx = \frac{1}{\|f\|^{p/p'}_{L^p(\Omega)}} \int_\Omega |f|^p \, dx = \frac{\|f\|^p_{L^p(\Omega)}}{\|f\|^{p/p'}_{L^p(\Omega)}} = \|f\|^{p - \frac{p}{p'}}_{L^p(\Omega)} = \|f\|_{L^p(\Omega)},$$

which proves the equality.

(v) Now, assume f is measurable but $f \notin L^p(\Omega)$. We set $\|f\|_{L^p(\Omega)} = +\infty$. Consider $g_{M,R}\colon \Omega \to \mathbb{R}$

defined by

$$g_{M,R} = f \mathbb{1}_{[|f| \leq M]} \mathbb{1}_{[|x| \leq R]}.$$

Then $g_{M,R} \in L^p(\Omega)$ since it is a bounded function with compact support and $\|g_{M,R}\|_{L^p(\Omega)} \to \infty$ as $M, R \to \infty$. Thus, for all $L > 0$, $\exists M_0, R_0$ such that $\|g_{M_0,R_0}\|_{L^p(\Omega)} > L$. By (iv), there exists $h_0 \in L^{p'}(\Omega)$ such that $\|h_0\|_{L^{p'}(\Omega)} \leq 1$ and

$$L \leq \|g_{M_0,R_0}\|_{L^p(\Omega)} = \int_\Omega g_{M_0,R_0} h_0 \, dx = \int_\Omega f \mathbb{1}_{[|f| \leq M_0]} \mathbb{1}_{[|x| \leq R_0]} h_0 \, dx = \int_\Omega f h_{M_0,R_0} \, dx,$$

where $h_{M_0,R_0} \colon \Omega \to \mathbb{R}$ defined by $h_{M_0,R_0} = h_0 \mathbb{1}_{[|f| \leq M_0]} \mathbb{1}_{[|x| \leq R_0]}$. Observe that $\|h_{M_0,R_0}\|_{L^{p'}(\Omega)} \leq \|h_0\|_{L^{p'}(\Omega)} \leq 1$ and

$$\int_\Omega f h_{M_0,R_0} \, d\lambda \geq L.$$

Thus,

$$\sup \left\{ \int_\Omega f g \, dx : g \in L^{p'}(\Omega), \|g\|_{L^{p'}(\Omega)} \leq 1 \right\} \geq L.$$

Since L is arbitrary, we get

$$\sup \left\{ \int_\Omega f g \, dx : g \in L^{p'}(\Omega), \|g\|_{L^{p'}(\Omega)} \leq 1 \right\} = \infty,$$

which finishes the exercise.