Serie 7

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Derive the following formula by expanding part of the integrand into a series and justify the term-by-term integration.

$$\int_0^\infty e^{-sx}\frac{\sin x}{x}\,dx = \arctan(1/s) \qquad \text{for } s > 1\,.$$

Solution: Define the sequence of functions $f_n(x) = \sum_{k=0}^n \frac{(-1)^k}{(2k+1)!}x^{2k}e^{-sx}$, for $n \geq 1$. Since $\sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ converges to $\sin(x)$ for all $x \in \mathbb{R}$, $f_n(x)$ converges pointwise to $f(x) = e^{-sx}\frac{\sin x}{x}$ for $x \in (0,\infty)$. The sequence $\{f_n\}_n$ is dominated by $F(x) = e^{-(s-1)x}$ on $[0,\infty)$:

$$|f_n(x)| \leq \sum_{k=0}^n \frac{1}{(2k+1)!}x^{2k}e^{-sx} \leq e^{-sx}\sum_{k=0}^n \frac{1}{(2k)!}x^{2k} \leq e^x e^{-sx} = e^{-(s-1)x}, \quad \forall x \in [0,\infty).$$

We know F is absolutely integrable on $[0,\infty)$ because $s > 1$, hence the dominated convergence theorem implies

$$\int_0^\infty e^{-sx}\frac{\sin x}{x}\,dx = \int_0^\infty f(x)dx = \lim_{n\to\infty}\int_0^\infty f_n(x)dx = \lim_{n\to\infty}\sum_{k=1}^n \frac{(-1)^k}{(2k+1)!}\int_0^\infty x^{2k}e^{-sx}dx$$
$$= \lim_{n\to\infty}\sum_{k=0}^n \frac{(-1)^k}{(2k+1)!}\frac{\Gamma(2k+1)}{s^{2k+1}} = \sum_{n=0}^\infty (-1)^n\frac{1}{(2n+1)s^{2n+1}} = \arctan(1/s)\,.$$

Exercise 2. Let $\Omega \subset \mathbb{R}^d$ be measurable. Prove that $L^\infty(\Omega)$ is complete. In other words, if $f_n\colon \Omega \to \mathbb{R}$, $n \in \mathbb{N}$ is a Cauchy sequence in $L^\infty(\Omega)$, then there is $f \in L^\infty(\Omega)$ such that

$$\lim_{n\to\infty}\|f_n - f\|_{L^\infty} = 0.$$

Hint: Note that $\{x \in \Omega : |f_n(x) - f_m(x)| \geq \|f_n - f_m\|_{L^\infty}\}$ has measure 0. Find a set E of measure 0 such that for any $x \in E^c$, $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy sequence. Then call the limit $f(x) = \lim_{n\to\infty} f_n(x)$, for all $x \in E^c$. We get a function f defined a.e. Finally show that $f \in L^\infty(\Omega)$ and $f_n \to f$ in L^∞.

Solution: For $\nu, \mu \in \mathbb{N}$ and $\nu \neq \mu$, we define

$$\begin{aligned} A_{\nu,\mu} &= \{x \in \Omega : |f_\nu(x) - f_\mu(x)| > \|f_\nu - f_\mu\|_{L^\infty}\}, \\ B_\nu &= \{x \in \Omega : |f_\nu(x)| \geq \|f_\nu\|_{L^\infty}\}\,. \end{aligned}$$

Since all these sets have Lebesgue measure 0, the set

$$E = \left\{ \bigcup_{\nu \neq \mu} A_{\nu,\mu} \right\} \cup \left\{ \bigcup_{\nu=1}^{\infty} B_\nu \right\}$$

has measure 0 as well (as the countable union of null sets). We claim that for $x \in E^c$ $\{f_\nu(x)\}_{\nu=1}^{\infty}$ is a Cauchy sequence. Indeed, $x \in E^c$ means that

$$x \in \bigcap_{\nu=1}^{\infty} A^c_{\nu,\mu},$$

and therefore $|f_\nu(x) - f_\mu(x)| \leq \|f_\nu - f_\mu\|_{L^\infty}$ for every $\nu, \mu \in \mathbb{N}$, hence Cauchy. For $x \in E^c$, we can therefore define $f(x)$ to be the limit of this Cauchy sequence, that is $f(x) = \lim_{\nu \to \infty} f_\nu(x)$, and for $x \in E$ we set $f(x)$ to an arbitrary value. We now claim that $f \in L^\infty(\Omega)$ and $f_\nu \to f$ in $L^\infty(\Omega)$.

Since f_ν is a Cauchy sequence in $L^\infty(\Omega)$, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|f_\nu - f_\mu\|_{L^\infty} \leq \varepsilon \quad \forall \nu, \mu \geq N\,.$$

Thus, for all $x \in E^c$ and if $\nu, \mu \geq N$ $|f_\nu(x) - f_\mu(x)| \leq \|f_\nu - f_\mu\|_{L^\infty} \leq \varepsilon$. Letting $\nu \to \infty$,

$$|f(x) - f_\mu(x)| = \lim_{\nu \to \infty} |f_\nu(x) - f_\mu(x)| \leq \varepsilon \quad \forall \mu \geq N.$$

Thus $|f - f_\mu| \leq \varepsilon$ a.e. and implies

$$|f| \leq |f - f_\mu| + |f_\mu| \leq |f_\mu| + \varepsilon \text{ a.e.}$$

As a consequence, we get $f \in L^\infty(\Omega)$. Moreover, since $|f - f_\mu| \leq \varepsilon$ a.e. for all $\mu \geq N$, we conclude

$$\|f - f_\mu\|_{L^\infty} \leq \varepsilon \quad \forall \mu \geq N,$$

which finishes the proof.

Exercise 3. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded and measurable set and let $f, f_n : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ for $n \in \mathbb{N}$ be measurable such that

(i) $f, f_n \in L^1(\Omega)$ for all $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} \|f_n - f\|_{L^\infty(\Omega)} = 0$.

Prove, using the dominated convergence theorem, that

$$\lim_{n\to\infty} \int_\Omega f_n \, dx = \int_\Omega f \, dx.$$

Compare this with the first theorem about permuting limit and integral seen during the first year analysis courses.

Solution: We first show that (ii) implies uniform convergence outside a set of measure 0 . Indeed, from (ii) we deduce that for every $k > 0$ there exists $N_k \in \mathbb{N}$ such that for any $n \geq N_k$ we have $\|f_n - f\|_{L^\infty(\Omega)} < \frac{1}{k}$. This implies that for all $n \geq N_k$ there exists $\Omega_{n,k} \subseteq \Omega$ with $\mathrm{m}(\Omega_{n,k}) = 0$ such that

$$|f_n(x) - f(x)| \leq \frac{1}{k} \quad \text{for all } x \in \Omega \setminus \Omega_{n,k}\,.$$

Set $\overline{\Omega} := \bigcup_{k\in\mathbb{N}} \bigcup_{n=N_k}^{\infty} \Omega_{n,k}$. Since countable union of null sets are null sets, we deduce that $\mathrm{m}(\overline{\Omega}) = 0$. By definition, it follows that $f_n \to f$ uniformly on $\Omega \setminus \overline{\Omega}$, in particular $f_n \to f$ pointwise on $\Omega \setminus \overline{\Omega}$.

Let now $\varepsilon > 0$. From the uniform convergence on $\Omega \setminus \overline{\Omega}$, we deduce that there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$

$$|f_n(x) - f(x)| \leq \varepsilon \quad \text{for all } x \in \Omega \setminus \overline{\Omega}\,. \tag{1}$$

We define the function

$$g := \max\left(|f_1|\,, |f_2|\,, ..., |f_{N-1}|\,, |f| + \varepsilon\right).$$

It is clear that g is a measurable and nonnegative function which satisfies pointwise

$$0 \leq g \leq \sum_{n=1}^{N-1} |f_n| + |f| + \varepsilon.$$

Moreover, since Ω is bounded, the constant function equal to ε belongs to $L^1(\Omega)$. The space $L^1(\Omega)$ being a vector space, we have

$$\sum_{n=1}^{N-1} |f_n| + |f| + \varepsilon \in L^1(\Omega).$$

By monotonicity of the integral, we get $g \in L^1(\Omega)$. Moreover, we claim that $|f_n(x)| \leq g(x)$ for every $x \in \Omega \setminus \overline{\Omega}$ and for all $n \in \mathbb{N}$ (in particular, the bound holds for almost every x). Indeed, if $x \in \Omega \setminus \overline{\Omega}$ and $n \leq N - 1$, the result follows from the definition of g. Otherwise, if $n \geq N$ we have by the triangular inequality and from the choice of N in (1) that for every $x \in \Omega \setminus \bar{\Omega}$

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \overset{(1)}{\leq} |f(x)| + \varepsilon \quad \leq g(x)\,.$$

Finally, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega f_n\,dx = \int_\Omega \lim_{n\to\infty} f_n\,dx = \int_\Omega f\,dx.$$

Exercise 4. We want to investigate the interaction of the L^∞-norm and continuous functions.

(i) Let $\Omega \subseteq \mathbb{R}^n$ open and assume that $E \subseteq \Omega$ is a zero measure set. Prove that $\Omega \setminus E$ is dense in Ω.

(ii) Let $f, g \in C(\mathbb{R}^n)$ such that $f = g$ almost everywhere. Show that $f = g$ everywhere.

(iii) Let $\Omega \subset \mathbb{R}^n$ open and $f \in C(\Omega)$. Prove

$$\|f\|_{L^\infty(\Omega)} = \|f\|_{C^0(\Omega)} = \sup_{x\in\Omega} |f(x)|.$$

Solution:

(i) Assume for a contradiction $\Omega \setminus E$ is not dense. Then, there is $x \in \Omega$ and $r > 0$ such that $B(x, r) \cap \Omega \cap (\Omega \setminus E) = \emptyset$. This implies $B(x, r) \cap \Omega \subset E$. Since Ω is open, we have that $\mathrm{m}(B(x, r) \cap \Omega) > 0$ and then, by the monotonicity of the Lebesgue measure

$$0 < \mathrm{m}(B(x, r) \cap \Omega) \leq \mathrm{m}(E),$$

which is a contradiction.

Remark : The inverse statement is not true. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$ but $\mathbb{R} \setminus \mathbb{Q}$ is not of measure 0.

(ii) Consider the function $h \in C(\mathbb{R}^n)$ defined by $h := f - g$. Clearly,

$$E := \{x \in \mathbb{R}^n : f(x) \neq g(x)\} = \{x \in \mathbb{R}^n : h(x) \neq 0\} = h^{-1}(\mathbb{R} \setminus \{0\})$$

is open as preimage of an open set through a continuous function. Since $f = g$ a.e.,

$$\mathrm{m}\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) = 0$$

and therefore $\mathbb{R}^n \setminus E$ is dense in $\mathbb{R}^n$ by (i). In addition $\mathbb{R}^n \setminus E$ is closed, thus $\mathbb{R}^n \setminus E = \overline{\mathbb{R}^n \setminus E} = \mathbb{R}^n$, which implies $E = \emptyset$. We conclude $f = g$ everywhere.

(iii) Let $f \in C(\Omega)$. We have the inequality

$$\|f\|_{L^\infty} = \inf\{\alpha \geq 0 : |f(x)| \leq \alpha \text{ a.e.}\} \leq \inf\{\alpha \geq 0 : |f(x)| \leq \alpha \text{ everywhere}\} = \sup_{x\in\Omega} |f(x)|.$$

Now, assume for a contradiction $\|f\|_{L^\infty} < \sup_{x\in\Omega} |f(x)|$ and consider two cases.

Case 1: $\sup_{x\in\Omega} |f(x)| = +\infty$.

In this case, for any constant $C > 0$, there exists $x_0 \in \Omega$ such that $|f(x_0)| \geq C$ and by assumption $\|f\|_{L^\infty} < +\infty$. Take $C = \|f\|_{L^\infty}+2$ and fix $x_0 \in \Omega$ such that $|f(x_0)| \geq \|f\|_{L^\infty}+2$. By continuity of f, there is $\delta > 0$ such that for all $x \in B(x_0, \delta)$, we have $|f(x) - f(x_0)| \leq 1$. Then, for all $x \in B(x_0, \delta)$,

$$|f(x)| \geq |f(x_0)| - |f(x_0) - f(x)| \geq \|f\|_{L^\infty} + 1.$$

Since $\mathrm{m}(B(x_0, \delta)) > 0$ and $|f| > \|f\|_{L^\infty}$ on $B(x_0, \delta)$, this gives a contradiction.

Case 2: $\sup_{x\in\Omega} |f(x)| < +\infty$.

Let $\varepsilon = \sup_{x\in\Omega}|f(x)| - \|f\|_{L^\infty} > 0$. By definition of the supremum, there exists $x_0 \in \Omega$ such that

$$|f(x_0)| \geq \sup_{x\in\Omega}|f(x)| - \frac{1}{3}\varepsilon.$$

Again, by the continuity of f, there is $\delta > 0$ such that for all $x \in B(x_0, \delta)$,

$$|f(x) - f(x_0)| \leq \frac{1}{3}\varepsilon.$$

In particular, for all $x \in B(x_0, \delta)$, we have by the choice of ε that

$$|f(x)| \geq |f(x_0)| - |f(x_0) - f(x)| \geq \sup_{x\in\Omega}|f(x)| - \frac{2}{3}\varepsilon = \|f\|_{L^\infty} + \frac{1}{3}\varepsilon,$$

Again, since $B(x_0, \delta)$ is not a measure zero set and ε is arbitrary, this is a contradiction.

Exercise 5 ($*$)**.** We show that there exist many non-measurable sets. More precisely, let $A \subseteq \mathbb{R}$ with $\mathrm{m}^*(A) > 0$. Show that then there exists $B \subseteq A$ such that B is not measurable.
Hint: Consider rational translations of the Vitali set V and use and prove the following
Claim: For any measurable set E such that $\mathrm{m}^*(E) > 0$ the difference set $D_E := \{x - y : x, y \in E\}$ contains an interval around the origin.

Solution: Recall the construction of the Vitali set. We call $A \subset \mathbb{R}$ a coset of $\mathbb{Q}$ if it takes the form $A = x + \mathbb{Q}$ for some $x \in \mathbb{R}$. Note that two cosets are either equal or disjoint and each coset has a nonempty intersection with $[0, 1]$. Using the axiom of choice we can pick an element in each of the cosets belonging to $[0, 1]$ and the union of these elements forms the Vitali set. Recall that the Vitali set V is not measurable. Moreover, if we define $V_q = q + V$ for all $q \in \mathbb{Q}$, we obtain

$$\bigcup_{q\in\mathbb{Q}} V_q = \mathbb{R}$$

Indeed, for any $x \in \mathbb{R}$, x belongs to some coset A of $\mathbb{Q}$. Hence there is $v \in V$, $q \in \mathbb{Q}$ such that $x = v + q$. Thus $x \in V_q$ and therefore $x \in \cup_{q\in\mathbb{Q}} V_q$. This proves $\bigcup_{q\in\mathbb{Q}} V_q = \mathbb{R}$. In addition $V_q \cap V_{q'} = \emptyset$ for all rationals $q \neq q'$. To prove this, assume by contradiction that some $x \in V_q \cap V_{q'}$. Then there exists $v, v' \in V$ such that $x = v + q = v' + q'$. It follows that v and v' belong to the same coset of $\mathbb{Q}$ and by construction of the Vitali set, $v = v'$. Thus, $q = q'$ which gives us a contradiction.

From subadditivity of the outer measure,

$$0 < \mathrm{m}^*(A) \leq \sum_{q\in\mathbb{Q}} \mathrm{m}^*(A \cap V_q).$$

Therefore, there exists a $q \in \mathbb{Q}$ such that $\mathrm{m}^*(A \cap V_q) > 0$. We will show that this set $A \cap V_q$ is non-measuable. To this aim, we need the following

Claim: *For any measurable set E such that $\mathrm{m}^*(E) > 0$ the difference set $D_E := \{x - y : x, y \in E\}$ contains an interval around the origin.*

Assume for a moment that we have already proved this claim. Notice that that the difference set D_{V_q} does not contain any rational number. (To prove it, assume for a contradiction that there is a non null rational number $r \in V_q$. Then there exists two distinct $u, v \in V$ such that

$$r = (u + q) - (v + q) = u - v,$$

which implies that u and v are in the same coset of $\mathbb{Q}$ and this contradicts the construction of the Vitali set.) In particular, the difference set $D_{A \cap V_q} \subset D_{V_q}$ cannot contain any interval around the origin and hence we deduce from the claim that $A \cap V_q$ is non-measurable.

We are left to prove the claim.

Proof of the claim: We can suppose without loss of generality that $\mathrm{m}^*(E) < \infty$. Indeed, it suffices to replace E by $E \cap [-n, n]$ with n large enough. Observe that if $F \subseteq E$, then $D_F \subseteq D_E$. Moreover, since $\mathrm{m}^*(E) > 0$ there exists a compact set $K \subseteq E$ such that $\mathrm{m}^*(K) > 0$ and then there is an open set U such that $K \subseteq U$ and

$$\mathrm{m}^*(U) < 2\,\mathrm{m}^*(K). \tag{2}$$

Now take $\delta := \mathrm{dist}(K, U^c) = \inf\{|x - y| : x \in K, y \in U^c\}$. Notice that since K is compact and U^c is closed, $\delta > 0$. Now we claim that for any $x \in (-\delta, \delta)$,

$$K \cap (x + K) \neq \emptyset.$$

Indeed, if we assume for a contradiction that for some $x \in (-\delta, \delta)$, $K \cap (x + K) = \emptyset$. Then, since $K \cup (x + K) \subset U$

$$\mathrm{m}^*(U) = \mathrm{m}(U) \geq \mathrm{m}(x + K) + \mathrm{m}(K) = 2\,\mathrm{m}(K) = 2\,\mathrm{m}^*(K),$$

which contradicts (2). Hence $(x + K) \cap K \neq \emptyset$, for all $x \in (-\delta, \delta)$. Thus,

$$(-\delta, \delta) \subseteq \{x - y : x, y \in K\} \subseteq \{x - y : x, y \in E\}.$$

This proves the claim and achieves the exercise.