Serie 6

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\Omega := (0,1)\times(0,1)$. Investigate the existence and equality of $\int_\Omega f\, d(x,y)$, $\int_0^1\int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1\int_0^1 f(x,y)\,dy\,dx$ for

(i) $f(x,y) := \frac{x^2-y^2}{(x^2+y^2)^2}$.

(ii) $f(x,y) := (1-xy)^{-a}$ for $a > 0$.

Compare your result with Fubini's Theorem.

Exercise 2. The *Dirichelet integral* is the improper integral defined by

$$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$

It is an example of existence of improper Riemann integral, but it is but **not** absolutely integrable in Lebesgue sense (therefore it is not Lebesgue integrable). In particular, prove that the following equality holds

$$\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}dx = \pi/2\,,$$

but the Lebesgue integral

$$\int_0^\infty |\frac{\sin x}{x}|dx = \infty$$

Remark 1. Notice that the Riemann integral is defined for **bounded** intervals and then extended to $\mathbb{R}$ with improper integrals. In class you studied that $f : [a,b] \to \mathbb{R}$ (with $a, b \in \mathbb{R}$, i.e. $a \neq -\infty$ and $b \neq \infty$) is Riemann integrable then it is Lebesgue integrable.

Exercise 3 (Layer cake representation)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable. For $\alpha > 0$, we set $E_\alpha := \{x \in \mathbb{R}^d : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|\,dx = \int_0^\infty \mathrm{m}(E_\alpha)\,d\alpha.$$

Exercise 4. From every L^p-convergent sequence $\{f_n\}_{n>0}$ $(1 \leq p < \infty)$, we can always extract a subsequence that converges pointwise almost everywhere. Yet, it might happen that the full sequence f_n nowhere converges pointwise. Here we discuss such an example. For all $n \in \mathbb{N}$ there exists a unique $m \in \mathbb{N}$ and a unique $j \in \{0, 1, ..., 2^m - 1\}$ such that $n = j + 2^m$. We define

$$I_n = [\frac{j}{2^m}, \frac{j+1}{2^m}) \qquad f_n(x) = \chi_{I_n}(x)$$

Observe that $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,2/4]}$, $f_6 = \chi_{[2/4,3/4]}$, $f_7 = \chi_{[3/4,1]}$, $f_8 = \chi_{[0,1/8]}, \cdots$.

(i) Show that f_n converges in $L^p(0,1)$ for $1 \leq p < \infty$.

(ii) Show that f_n converges pointwise nowhere on $[0,1)$.

(iii) Find a subsequence of f_n which converges pointwise a.e. on $[0,1)$.

Exercise 5. Let $\Omega \subseteq \mathbb{R}^n$ be measurable. Show that if $\Omega \subset \mathbb{R}^n$ is bounded and if $f \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)} .$$

Hint: Show the following inequalities

$$\limsup_{p\to\infty} \|f\|_{L^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)},$$
$$\liminf_{p\to\infty} \|f\|_{L^p(\Omega)} \geq \|f\|_{L^\infty(\Omega)} - \varepsilon \quad \forall\, \varepsilon > 0 .$$

For the second inequality, study the set $A_\varepsilon := \{x \in \Omega : \ |f(x)| \geq \|f\|_{L^\infty} - \varepsilon\}$.

Exercise 6. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions.

(i) Assuming the result of Exercise 8 of Series 6, prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2n}$ (as a function of $(x,y) \in \mathbb{R}^{2n}$).

(ii) Show that if f and g are integrable on $\mathbb{R}^n$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2n}$ (as a function of $(x,y) \in \mathbb{R}^{2n}$).

(iii) We define the convolution of two integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, dy.$$

Show that $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$ (that is, $y \mapsto f(x-y)g(y)$ is an integrable function on $\mathbb{R}^n$ for a.e. $x \in \mathbb{R}^n$ fixed).

(iv) Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

with equality if f and g are non-negative.

(v) Recall that the Fourier transform $\hat{f}$ of an integrable function $f \in L^1(\mathbb{R}^n)$ defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

for $\xi \in \mathbb{R}^n$. Check first that $\hat{f}$ is bounded and continuous function of ξ. Then prove that for $f, g \in L^1(\mathbb{R}^n)$ integrable, one has

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \qquad \forall \xi \in \mathbb{R}^n .$$

Exercise 7. Consider the convolution of two measurable functions $f, g : \mathbb{R}^d \to \mathbb{R}$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y) g(y) \, dy .$$

Recall the properties of the convolution that we have established in the previous exercise.

(i) Show that $f * g$ is well-defined for every $x \in \mathbb{R}^d$ and that $f * g$ is uniformly continuous, if f is integrable and g is bounded.

(ii) If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Exercise 8 ($\star$)**.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Prove that the function $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(x, y) = f(x - y)$, ($x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$) is measurable.