Serie 6

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\Omega := (0,1)\times(0,1)$. Investigate the existence and equality of $\int_\Omega f\, d(x,y)$, $\int_0^1\int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1\int_0^1 f(x,y)\,dy\,dx$ for

(i) $f(x,y) := \frac{x^2-y^2}{(x^2+y^2)^2}$.

(ii) $f(x,y) := (1-xy)^{-a}$ for $a>0$.

Compare your result with Fubini's Theorem.

Solution:

(i) We have

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1\int_0^1 \partial_y\left(\frac{y}{x^2+y^2}\right)dy\,dx = \int_0^1 \frac{1}{x^2+1}\,dx = [\arctan(x)]_0^1 = \frac{\pi}{4}$$

and

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1\int_0^1 \partial_x\left(\frac{-x}{x^2+y^2}\right)dx\,dy = \int_0^1 \frac{-1}{y^2+1}\,dy = [-\arctan(y)]_0^1 = -\frac{\pi}{4}$$

This computation does not contradict Fubini as $f \notin L^1(\Omega)$. Indeed, we compute by Tonelli:

$$\begin{aligned}
\int_0^1\int_0^1 |f(x,y)|\,dy\,dx &= \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx + \int_0^1\int_x^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx \\
&= \int_0^1\int_0^x \partial_y\left(\frac{y}{x^2+y^2}\right)dy\,dx + \int_0^1\int_x^1 \partial_y\left(\frac{-y}{x^2+y^2}\right)dy\,dx \\
&= \int_0^1 \left[\frac{y}{x^2+y^2}\right]_0^x dx + \int_0^1 \left[\frac{-y}{x^2+y^2}\right]_x^1 dx \\
&= \int_0^1 \frac{x}{2x^2} - \frac{1}{1+x^2} + \frac{x}{2x^2}\,dx = \int_0^1 \frac{1}{x} - \frac{1}{1+x^2}\,dx = \infty.
\end{aligned}$$

(ii) Let $f_a(x,y) = (1-xy)^{-a}$. Note that $f_a(x,y)$ is positive on Ω so $\int_\Omega f_a(x,y)d(x,y) = \int_0^1\int_0^1 f_a(x,y)\,dy\,dx = \int_0^1\int_0^1 f_a(x,y)\,dx\,dy$ by Tonelli's theorem. We show that $\int_\Omega f_a(x,y)d(x,y)$ is finite if and only if $0 < a < 2$. Let's consider two cases: $a = 1$ and $a \neq 1$.

- Case $a = 1$: In this case the integral is finite and we can actually compute it.

$$\begin{aligned}
\int_0^1 \left(\int_0^1 \frac{dx}{1-xy}\right) dy &= \int_0^1 \left(-\frac{1}{y}\int_1^{1-y}\frac{ds}{s}\right) dy \quad \text{by setting } s = 1-xy, ds = -y\,dx \\
&= \int_0^1 \frac{1}{y}[\log(s)]_{1-y}^1\,dy = \int_0^1 \frac{-\log(1-y)}{y}\,dy \\
&= \int_0^1 \sum_{k=1}^\infty \frac{y^{k-1}}{k}\,dy \quad \text{by expanding } \log(1-y) \text{ into its Maclaurin series} \\
&= \sum_{k=1}^\infty \int_0^1 \frac{y^{k-1}}{k}\,dy = \sum_{k=1}^\infty \left[\frac{1}{k^2}y^k\right]_0^1 dy = \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6},
\end{aligned}$$

  where we interchanged the serie and the integral by either monotone convergence (the serie has positive terms) or by uniform convergence of the Maclaurin serie.

- Case $a \neq 1$: We have

$$\begin{aligned}
\int_0^1 \left(\int_0^1 \frac{dx}{(1-xy)^a}\right) dy &= \int_0^1 -\frac{1}{y}\left(\int_1^{1-y}\frac{ds}{s^a}\right) dy \\
&= \int_0^1 -\frac{1}{y}\left[\frac{s^{1-a}}{1-a}\right]_1^{1-y} dy \\
&= \int_0^1 -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a}\,dy.
\end{aligned}$$

  Observe that the integrand $y \mapsto -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a}$ is continuous and bounded (by $\frac{1}{1-a}$) on $(0,1)$ if $0 < a < 1$, and thus has a finite integral.

  So assume $a > 1$. In this case, observe that

$$(1-y)^{1-a} \leq -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a} \leq \frac{1}{a-1}(1-y)^{1-a}, \qquad \forall y \in (0,1).$$

  Note that the integral $\int_0^1(1-y)^{1-a}dy$ is finite if and only if $a-1 < 1$, i.e., $a < 2$. Hence, we conclude that the integral $\int_0^1 -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a}\,dy$ is finite if and only if $a < 2$.

We conclude that f_a is absolutely integrable if and only if $0 < a < 2$. Since f is absolutely integrable, we can apply Fubini's theorem, which is consistent with our computations.

Exercise 2. The *Dirichelet integral* is the improper integral defined by

$$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$

It is an example of existence of improper Riemann integral, but it is but **not** absolutely integrable in Lebesgue sense (therefore it is not Lebesgue integrable). In particular, prove that the following equality holds

$$\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}dx = \pi/2\,,$$

but the Lebesgue integral

$$\int_0^\infty |\frac{\sin x}{x}|dx = \infty$$

Remark 1. Notice that the Riemann integral is defined for **bounded** intervals and then extended to $\mathbb{R}$ with improper integrals. In class you studied that $f:[a,b]\to\mathbb{R}$ (with $a,b\in\mathbb{R}$, i.e. $a\neq-\infty$ and $b\neq\infty$) is Riemann integrable then it is Lebesgue integrable.

Solution:

You might have already encountered this integral in other analysis courses and you were maybe asked to show that the function $x\mapsto\frac{\sin(x)}{x}$ is *Riemann-integrable* but not *absolutely integrable.* Let's recall the proofs of theses two properties :

- The function $x\mapsto\frac{\sin(x)}{x}$ is Riemann-integrable on $[0,\infty)$ because it can be extended by continuity at 0, and in particular it is Riemann-integrable on $[0,\pi]$. Also, $\forall t\geq\pi$

$$\int_\pi^t \frac{\sin(x)}{x}\,dx \underset{\text{IBP}}{=} \left[\frac{-\cos(x)}{x}\right]_\pi^t - \int_\pi^t \frac{\cos(x)}{x^2}\,dx = \frac{-\cos(t)}{t} - \frac{1}{\pi} - \int_\pi^t \frac{\cos(x)}{x^2}\,dx.$$

  We know that the integral $\int_\pi^\infty \frac{\cos(x)}{x^2}\,dx$ exists and equals $\lim_{t\to\infty}\int_\pi^t \frac{\cos(x)}{x^2}\,dx$ because $\left|\frac{\cos(x)}{x^2}\right|$ is dominated by $\frac{1}{x^2}$. We conclude that $\int_\pi^t \frac{\sin(x)}{x}\,dx$ converges as $t\to\infty$.

- The function is not not absolutely integrable because

$$\begin{aligned}\int_0^\infty \left|\frac{\sin(x)}{x}\right|\,dx &= \lim_{n\to\infty}\sum_{k=0}^n \int_{k\pi}^{(k+1)\pi}\left|\frac{\sin(x)}{x}\right|\,dx\\ &\geq \lim_{n\to\infty}\sum_{k=0}^n \frac{1}{(k+1)\pi}\underbrace{\int_{k\pi}^{(k+1)\pi}|\sin(x)|\,dx}_{=2}\\ &= \frac{2}{\pi}\sum_{k=1}^\infty\frac{1}{k} = \infty.\end{aligned}$$

  For the first equality, we implicitly used the monotone convergence theorem.

To compute the value of the integral, we use Fubini's theorem, which can only be applied to absolutely integrable functions. We therefore restrict the domain of integration to a bounded one to understand the integral as a Lebesgue integral:

$$\int_0^t \frac{\sin x}{x}\,dx = \int_0^t \sin x\int_0^\infty e^{-ax}\,da\,dx \underset{\text{Fubini}}{=} \int_0^\infty \underbrace{\int_0^t \sin x e^{-ax}\,dx}_{=:I_t(a)}\,da$$

Now the inner integral $I_t(a)$ is taken on a bounded domain so we can again re-interpret it as a Riemann-integral and use integration by parts:

$$I_t(a) = \left[-\cos(x)e^{-ax} - a\sin(x)e^{-ax}\right]_0^t - a^2 I_t(a) \Longrightarrow I_t(a) = \frac{1-e^{-at}(\cos(t)+a\sin(t))}{1+a^2}$$

Notice that the function I_t is dominated by $a \mapsto \frac{2}{1+a^2}$, which is Lebesgue integable, and that $\lim_{t\to\infty} I_t(a) = \frac{1}{1+a^2}$. We can therefore use the dominated convergence theorem to conclude:

$$\lim_{t\to\infty} \int_0^t \frac{\sin(x)}{x}\, dx = \lim_{t\to\infty} \int_0^\infty I_t(a) = \int_0^\infty \frac{da}{1+a^2} = \frac{\pi}{2}.$$

Exercise 3 (Layer cake representation)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable. For $\alpha > 0$, we set $E_\alpha := \{x \in \mathbb{R}^d : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty \mathrm{m}(E_\alpha)\, d\alpha.$$

Solution: First, notice that

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_{\mathbb{R}^d} \int_0^{|f(x)|} 1\, d\alpha\, dx = \int_{\mathbb{R}^d} \int_0^\infty \mathbb{1}_{[\alpha<|f(x)|]}(x,\alpha)\, d\alpha\, dx$$

where

$$\mathbb{1}_{[\alpha<|f(x)|]}(x,\alpha) = \begin{cases} 1 & \text{if } \alpha < |f(x)|; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{1}_{[\alpha<|f(x)|]} \geq 0$ is a non-negative and measurable function (here we use the measurability of f), we can use Tonelli's theorem to commute the integrals. In other words, we get

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{[\alpha<|f(x)|]}(x,\alpha)\, dx\, d\alpha = \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{[x\in E_\alpha]}(x,\alpha)\, dx\, d\alpha = \int_0^\infty \mathrm{m}(E_\alpha)\, d\alpha.$$

Exercise 4. From every L^p-convergent sequence $\{f_n\}_{n>0}$ $(1 \leq p < \infty)$, we can always extract a subsequence that converges pointwise almost everywhere. Yet, it might happen that the full sequence f_n nowhere converges pointwise. Here we discuss such an example. For all $n \in \mathbb{N}$ there exists a unique $m \in \mathbb{N}$ and a unique $j \in \{0, 1, ..., 2^m - 1\}$ such that $n = j + 2^m$. We define

$$I_n = [\frac{j}{2^m}, \frac{j+1}{2^m}) \qquad f_n(x) = \chi_{I_n}(x)$$

Observe that $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,2/4]}$, $f_6 = \chi_{[2/4,3/4]}$, $f_7 = \chi_{[3/4,1]}$, $f_8 = \chi_{[0,1/8]}, \cdots$.

(i) Show that f_n converges in $L^p(0,1)$ for $1 \leq p < \infty$.

(ii) Show that f_n converges pointwise nowhere on $[0,1)$.

(iii) Find a subsequence of f_n which converges pointwise a.e. on $[0,1)$.

Solution:

(i) With $1 \le p < \infty$ we have

$$\|f_\nu\|_{L^p} = \mathrm{m}(I_\nu)^{1/p} = \frac{1}{2^{h(\nu)/p}} \to 0 \text{ as } \nu \to \infty$$

and thus $f_\nu \to 0$ in $L^p(0,1)$ as $\nu \to \infty$.

(ii) For all $n \in \mathbb{N}$, the family $\{I_{2^n}, I_{2^n+1}, \dots, I_{2^{n+1}-1}\}$ covers $[0,1]$. It follows that for all $x \in [0,1]$ and $k \in \mathbb{N}$, there is $\nu_k^0, \nu_k^1 > k$ such that

$$x \notin I_{\nu_k^0} \quad \text{and} \quad x \in I_{\nu_k^1}.$$

Thus, for all $x \in [0,1]$, there are two subsequences (depending on x) of $\{f_\nu\}$, given by $\{f_k^0\}_{k=1}^\infty$ and $\{f_k^1\}_{k=1}^\infty$, such that

$$\lim_{k\to\infty} f_k^0(x) = 0 \quad \text{and} \quad \lim_{k\to\infty} f_k^1(x) = 1\,.$$

We deduce that $\lim_{\nu\to\infty} f_\nu(x)$ does not exist.

(iii) Consider the subsequence $\tilde{f}_k := f_{2^k} = \chi_{[0,2^{-k}]}$ for $k \ge 0$. Then, for all $x > 0$, there is $k \ge 1$ such that $x > 2^{-k}$, hence $\lim_{k\to\infty} \tilde{f}_k(x) = 0$. However, if $x = 0$, we have $\lim_{k\to\infty} \tilde{f}_k(0) = 1$. We conclude that $\tilde{f}_k \to 0$ a.e.

Exercise 5. Let $\Omega \subseteq \mathbb{R}^n$ be measurable. Show that if $\Omega \subset \mathbb{R}^n$ is bounded and if $f \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}\,.$$

Hint: Show the following inequalities

$$\limsup_{p\to\infty} \|f\|_{L^p(\Omega)} \le \|f\|_{L^\infty(\Omega)},$$
$$\liminf_{p\to\infty} \|f\|_{L^p(\Omega)} \ge \|f\|_{L^\infty(\Omega)} - \varepsilon \quad \forall\, \varepsilon > 0\,.$$

For the second inequality, study the set $A_\varepsilon := \{x \in \Omega : \ |f(x)| \ge \|f\|_{L^\infty} - \varepsilon\}$.

Solution: Let $f \in L^\infty(\Omega)$. Recalling from the Lectures that

$$\|f\|_{L^p} \le |\Omega|^{1/p} \|f\|_{L^\infty}, \tag{1}$$

we get, since $|\Omega|^{1/p} \to 1$ as $p \to \infty$, that

$$\limsup_{p\to\infty} \|f\|_{L^p} \le \|f\|_{L^\infty}. \tag{2}$$

In order to show the reverse inequality, fix $0 < \varepsilon < \|f\|_{L^\infty}$ and consider the set

$$A_\varepsilon := \{x \in \Omega : \ |f(x)| \geq \|f\|_{L^\infty} - \varepsilon\}.$$

By the definition of the essential supremum, it is clear that $\mathrm{m}(A_\varepsilon) > 0$. Thus,

$$\int_\Omega |f|^p \, dx \geq \int_{A_\varepsilon} |f|^p \, dx \geq \mathrm{m}(A_\varepsilon)(\|f\|_{L^\infty} - \varepsilon)^p > 0,$$

and therefore, taking the p-th root,

$$\|f\|_{L^p} \geq \mathrm{m}(A_\varepsilon)^{1/p}(\|f\|_{L^\infty} - \varepsilon).$$

Since $\mathrm{m}(A_\varepsilon) > 0$ we have $\mathrm{m}(A_\varepsilon)^{1/p} \to 1$ as $p \to \infty$, hence taking the limit $p \to \infty$, we obtain

$$\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_\infty - \varepsilon.$$

Since ε was arbitrary, we deduce

$$\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}. \tag{3}$$

Exercise 6. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions.

(i) Assuming the result of Exercise 8 of Series 6, prove that $f(x - y)g(y)$ is measurable on $\mathbb{R}^{2n}$ (as a function of $(x, y) \in \mathbb{R}^{2n}$).

(ii) Show that if f and g are integrable on $\mathbb{R}^n$, then $f(x - y)g(y)$ is integrable on $\mathbb{R}^{2n}$ (as a function of $(x, y) \in \mathbb{R}^{2n}$).

(iii) We define the convolution of two integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) \, dy.$$

Show that $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$ (that is, $y \mapsto f(x - y)g(y)$ is an integrable function on $\mathbb{R}^n$ for a.e. $x \in \mathbb{R}^n$ fixed).

(iv) Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

with equality if f and g are non-negative.

(v) Recall that the Fourier transform $\hat{f}$ of an integrable function $f \in L^1(\mathbb{R}^n)$ defined as

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi} \, dx$$

for $\xi \in \mathbb{R}^n$. Check first that $\hat{f}$ is bounded and continuous function of ξ. Then prove that for $f, g \in L^1(\mathbb{R}^n)$ integrable, one has

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \qquad \forall \xi \in \mathbb{R}^n .$$

Solution:

(i) Using the result of Exercise 8, we know that the function $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(x, y) = f(x - y)$ (as a function of $(x, y) \in \mathbb{R}^{2n}$) is measurable. Now we prove that the function $G : \mathbb{R}^{2n} \to \mathbb{R}$ given by $G(x, y) = g(y)$ (as function of $(x, y) \in \mathbb{R}^{2n}$) is measurable. Indeed, notice

$$\{(x, y) \in \mathbb{R}^{2n} : G(x, y) > \alpha\} = \{(x, y) \in \mathbb{R}^{2n} : g(y) > \alpha\} = \mathbb{R}^n \times \{y \in \mathbb{R}^n : g(y) > \alpha\}$$

which is measurable since it is the product of two measurable sets. Finally, the fact that $f(x - y)g(y)$ is a measurable function of $(x, y) \in \mathbb{R}^{2n}$ follows from the fact $f(x - y)g(y) = F(x, y)G(x, y)$ and that the product of two measurable functions is measurable.

(ii) Since the function $f(x - y)g(y)$ is measurable, using Tonelli's theorem

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} |f(x - y)g(y)| \, d(x, y) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)||g(y)| \, dx \, dy \\
&= \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(x - y)| \, dx \, dy \\
&= \|f\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |g(y)| \, dy \\
&= \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.
\end{aligned}$$

(iii) By (ii), $f(x - y)g(y)$ is integrable on $\mathbb{R}^{2n}$ and thus, using Fubini's theorem, we have

$$\int_{\mathbb{R}^n} |f(x - y)g(y)| dy < \infty \qquad \text{for a.e } x \in \mathbb{R}^n$$

and hence $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$.

(iv) We have (similar to the computation in (ii))

$$\begin{aligned}
\int_{\mathbb{R}^n} |(f * g)(x)| \, dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x - y)g(y) \, dy \right| \, dx &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)g(y)| \, dy \, dx \\
&\leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},
\end{aligned}$$

and thus

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

If f and g are non-negative, then, again by Fubini,

$$\begin{aligned}\|f*g\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right|\,dx &&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,dx\\ &= \int_{\mathbb{R}^n} g(y)\left(\int_{\mathbb{R}^n} f(x-y)\,dx\right)\,dy &&= \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\,,\end{aligned}$$

where the last equality uses again the non-negativity of f and g together with a change of variables.

(v) Let $f \in L^1(\mathbb{R}^n)$. We have for $\xi \in \mathbb{R}^n$

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)e^{-2\pi i x\cdot\xi}|\,dx \leq \|f\|_{L^1}\|\phi_\xi\|_{L^\infty} = \|f\|_{L^1},$$

where, $\phi_\xi(x) = e^{-2\pi i x\cdot\xi}$, hence $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$, which shows that $\hat{f}$ is a bounded function. Now for the continuity, let $\xi_n \to \xi$. Then

- $f(x)e^{-2\pi i x\cdot\xi_n} \to f(x)e^{-2\pi i x\cdot\xi}$ for every $x \in \mathbb{R}^n$,
- $|f(x)e^{-2\pi i x\cdot\xi_n}| \leq |f(x)|$ for every $x \in \mathbb{R}^n$,
- $|f| \in L^1(\mathbb{R}^n)$ by assumption.

We can thus apply the dominated convergence theorem to deduce

$$\lim_{n\to\infty}\hat{f}(\xi_n) = \lim_{n\to\infty}\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi_n}\,dx = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx = \hat{f}(\xi),$$

which proves continuity. Hence $\hat{f}$ is a bounded continuous function. Since f, g are integrable $f*g \in L^1(\mathbb{R}^n)$ by (iv) and hence, by the above discussion, the Fourier transforms of f, g and $f*g$ is well-defined and a continuous and bounded function. In particular, they are defined for every $\xi \in \mathbb{R}^n$ and not only almost everywhere. Finally, by Fubini

$$\begin{aligned}\widehat{(f*g)}(\xi) &= \int_{\mathbb{R}^n} (f*g)(x)e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right\}e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)e^{-2\pi i x\cdot\xi}\,dy\,dx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)e^{-2\pi i(x-y)\cdot\xi}g(y)e^{-2\pi i y\cdot\xi}\,dx\,dy\\ &= \int_{\mathbb{R}^n} g(y)e^{-2\pi i y\cdot\xi}\int_{\mathbb{R}^n} f(x-y)e^{-2\pi i(x-y)\cdot\xi}\,dx\,dy\\ &= \int_{\mathbb{R}^n} g(y)e^{-2\pi i y\cdot\xi}\hat{f}(\xi)\,dy = \hat{f}(\xi)\hat{g}(\xi).\end{aligned}$$

Exercise 7. Consider the convolution of two measurable functions $f, g : \mathbb{R}^d \to \mathbb{R}$ defined by

$$(f*g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy\,.$$

Recall the properties of the convolution that we have established in the previous exercise.

(i) Show that $f * g$ is well-defined for every $x \in \mathbb{R}^d$ and that $f * g$ is uniformly continuous, if f is integrable and g is bounded.

(ii) If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Solution:

(i) First of all, observe that the convolution is well-defined. Indeed, for every $x \in \mathbb{R}^d$ fixed, $y \mapsto f(x-y)g(y)$ is measurable on $\mathbb{R}^d$ (as a product of measurable functions) and hence

$$|(f * g)(x)| \leq \int_{\mathbb{R}^d} |f(x-y)||g(y)|\, dy \leq ||g||_{L^\infty(\mathbb{R}^d)}||f||_{L^1(\mathbb{R}^d)}$$

is well-defined. We now show the uniform continuity. To this aim, we define for $h \in \mathbb{R}^d$ the function $f_h : \mathbb{R}^d \to \mathbb{R}$ by $f_h(x) = f(x+h)$. Recall that

$$\lim_{h\to 0} \|f_h - f\|_{L^1(\mathbb{R}^d)} = 0$$

due to Exercise 2. This means in particular that for any $\varepsilon > 0$, there exist a $\delta > 0$ such that for all $|h| < \delta$, we have

$$\|f_h - f\|_{L^1(\mathbb{R}^d)} < \frac{\varepsilon}{\|g\|_{L^\infty(\mathbb{R}^d)}}.$$

Thus for all $x, z \in \mathbb{R}^d$ such that $|z - x| < \delta$, we have

$$\begin{aligned}(f * g)(x) - (f * g)(z) &= \int_{\mathbb{R}^d} [f(x-y) - f(z-y)]g(y)\, dy \\ &= \int_{\mathbb{R}^d} [f(x-y) - f(x-y+(z-x))]g(y)\, dy \\ &= \int_{\mathbb{R}^d} [f(x-y) - f_{z-x}(x-y)]g(y)\, dy.\end{aligned}$$

Using Hölder's inequality on the right-hand side, we obtain

$$|(f * g)(x) - (f * g)(z)| \leq \|f - f_{z-x}\|_{L^1}\|g\|_{L^\infty} < \varepsilon$$

for all $|x - z| < \delta$. This proves that $(f * g)$ is uniformly continuous.

(ii) Note that we know from Exercise 6 of Serie 6 that $\|(f*g)\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^1(\mathbb{R}^d)} < +\infty$ and therefore $f*g$ is integrable. Therefore it suffices, due to (i), to prove that any integrable function $h : \mathbb{R}^d \to \mathbb{R}$ that is uniformly continuous satisfies $|h(x)| \to 0$ as $|x| \to \infty$. We already proved this in Exercise 3 (iii) of Serie 5, in one dimension. The exact same argument can be applied here as well. It suffices to replace intervals by d-dimensional balls.

Exercise 8 ($\star$)**.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Prove that the function $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(x, y) = f(x - y)$, $(x \in \mathbb{R}^n,\ y \in \mathbb{R}^n)$ is measurable.

Solution: For any $\alpha \in \mathbb{R}$ we will show that

$$\widehat{E}_\alpha = \{(x,y) \in \mathbb{R}^{2n} : F(x,y) < \alpha\}$$

is measurable. Then for any set $\mathcal{A} \subset \mathbb{R}^n$, we define the set $\widetilde{\mathcal{A}}$ by

$$\widetilde{\mathcal{A}} := \{(x,y) \in \mathbb{R}^{2n} : x - y \in \mathcal{A}\}$$

Define

$$E_\alpha = \{z \in \mathbb{R}^n : f(z) < \alpha\}$$

and note that $\widehat{E}_\alpha = \widetilde{E}_\alpha = \{(x,y) : x - y \in E_\alpha\}$, so that is suffices to show that $\widetilde{E}_\alpha$ is measurable.
Step 1: As a preliminary result, we show that any measurable set can be written as the union of a countable union and intersection of closed sets and a measure 0 set.
Let E be a measurable set and consider E^c. We first want to show that there exists $\mathcal{O}$, a countable union and intersection of open sets, such that

$$\mathrm{m}(\mathcal{O} \setminus E^c) = 0\,. \tag{4}$$

To show (4), assume first that $\mathrm{m}(E^c) = \mathrm{m}^*(E^c) < +\infty$. By definition of the outer measure, there exists for any $k \in \mathbb{N}$ an open set O_k such that $E^c \subset O_k$ and $\mathrm{m}^*(O_k \setminus E^c) \leq 1/k$. Clearly, $E^c \subset \bigcap_{k=1}^\infty O_k$ and

$$\mathrm{m}^*\left(\bigcap_{k=1}^\infty O_k \setminus E^c\right) \leq \mathrm{m}^*(O_j \setminus E^c) \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

We conclude $\mathrm{m}(\bigcap_{k=1}^\infty O_k \setminus E^c) = 0$ and the desired set is therefore given by $\mathcal{O} := \bigcap_{k=1}^\infty O_k$.

If now $\mathrm{m}(E^c) = \infty$, then we consider $F_n := E^c \cap B(0,n)$. Since F_n is bounded, $\mathrm{m}(F_n) < +\infty$ and by the above procedure, we find for every $n \in \mathbb{N}$ a collection of open sets $\{O_{n,k}\}_{k=1}^\infty$ such that

$$\mathrm{m}\left(\bigcap_{k=1}^\infty O_{n,k} \setminus F_n\right) = 0\,.$$

We use that by construction $E^c = \bigcup_{n=1}^\infty F_n$ and we conclude by subadditivity that

$$\mathrm{m}\left(\left[\bigcup_{n=1}^\infty \bigcap_{k=1}^\infty O_{n,k}\right] \setminus E^c\right) \leq \sum_{n=1}^\infty \mathrm{m}\left(\bigcap_{k=1}^\infty O_{n,k} \setminus F_n\right) = 0\,.$$

The desired set is therefore given by $\mathcal{O} := \bigcup_{n=1}^\infty \bigcap_{k=1}^\infty O_{n,k}$. This finishes the proof of (4).

We now define $Z := \mathcal{O} \setminus E^c$, where $\mathcal{O}$ is given by (4). By construction, we have $E^c = \mathcal{O} \setminus Z$ and by taking complements on both sides, we obtain

$$E = \mathcal{O}^c \cup Z\,,$$

where $\mathcal{O}^c = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty O_{n,k}^c$ with $O_{n,k}^c$ closed (as complements of open sets).
Step 2: We show that for any open set $\mathcal{O} \subset \mathbb{R}^n$, the set $\widetilde{\mathcal{O}}$ is open and for every closed set $\mathcal{C} \subset \mathbb{R}^n$ the set $\widetilde{\mathcal{C}}$ is closed.
Indeed, let $\mathcal{O}$ open and $z \in \widetilde{\mathcal{O}}$, then z writes $z = (x,y)$, where $x - y \in \mathcal{O}$. Since $\mathcal{O}$ is open, let

$r > 0$ be such that $B(x-y,r) \subset \mathcal{O}$. All the norms being equivalent in finite dimension, we define the norm on $\mathbb{R}^{2n}$ as $\|(u,v)\|_{\mathbb{R}^{2n}} = \max(\|u\|_{\mathbb{R}^n}, \|v\|_{\mathbb{R}^n})$. Then, for any $h = (a,b) \in B\left(z, \frac{r}{2}\right)$, by the triangular inequality,

$$\|(a-b)-(x-y)\|_{\mathbb{R}^n} \leq \|a-x\|_{\mathbb{R}^n} + \|b-y\|_{\mathbb{R}^n} \leq 2\|(x,y)-(a,b)\|_{\mathbb{R}^{2n}} \leq r\,,$$

therefore $a-b$ is in $\mathcal{O}$ and h is in $\widetilde{\mathcal{O}}$. This proves that $\widetilde{\mathcal{O}}$ is open. The claim about closed sets follows by writing a closed set $\mathcal{C}$ as $\mathcal{C} = \mathbb{R}^n \setminus \mathcal{O}$ for some open set $\mathcal{O}$.

Step 3: We prove that if $\mathcal{A}$ can be written as a countable union and intersection of closed sets, then the same holds for $\widetilde{\mathcal{A}}$.
Indeed, assume that there is a countable family $\{\mathcal{C}_{n,k}\}_{n,k\in\mathbb{N}}$ of closed sets such that

$$\mathcal{A} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{C}_{n,k}.$$

Then clearly,

$$\widetilde{\mathcal{A}} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \widetilde{\mathcal{C}}_{n,k},$$

and due to the previous step every $\widetilde{\mathcal{C}}_{n,k} \subset \mathbb{R}^{2n}$ is a closed set.

Step 4: We prove that $\widetilde{E}_\alpha$ is measurable.
Since E_α is measurable, there is a set $\mathcal{A}$ that can be written as a countable union and intersection of closed sets $\mathcal{A}$ and measure 0 set $\mathcal{Z}$ such that $E_\alpha = \mathcal{A} \cup \mathcal{Z}$ and therefore $\widetilde{E}_\alpha = \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{Z}}$. Due to the previous step combined with the fact that any countable union and intersection of measurable sets is measurable, it suffices to prove that $\widetilde{\mathcal{Z}}$ is measurable. Actually, we will prove that $\mathrm{m}^*(\widetilde{\mathcal{Z}}) = 0$ and therefore it is measurable. For any $n \in \mathbb{N}$, let $\mathcal{O}^n$ be an open set such that $\mathcal{Z} \subset \mathcal{O}^n$ such that $\mathrm{m}(\mathcal{O}^n) < 1/n$. Then define,

$$B_k = \{(x,y) \in \mathbb{R}^{2n} : |y| \leq k\} = \mathbb{R}^n \times B(y,k),$$

and put $\widetilde{\mathcal{Z}}_k := \widetilde{\mathcal{Z}} \cap B_k$ and $\widehat{\mathcal{O}}_k^n = \widetilde{\mathcal{O}}^n \cap B_k$. Note that $\widetilde{\mathcal{Z}}_k \subset \widehat{\mathcal{O}}_k^n$. Now we will prove that

$$\mathrm{m}(\widehat{\mathcal{O}}_k^n) = \mathrm{m}(\mathcal{O}^n)\,\mathrm{m}(B_k).$$

First note that $\chi_{\widehat{\mathcal{O}}_k^n}(x,y) = \chi_{\widetilde{\mathcal{O}^n}\cap B_k}(x,y) = \chi_{\mathcal{O}^n}(x-y)\chi_{B_k}(y)$, and therefore

$$\mathrm{m}(\widehat{\mathcal{O}}_k^n) = \int_{\mathbb{R}^{2n}} \chi_{\mathcal{O}}(x-y)\chi_{B_k}(y)d(x,y) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \chi_{\mathcal{O}^n}(x-y)\,dx\right)\chi_{B_k}(y)\,dy = \mathrm{m}(\mathcal{O}^n)\,\mathrm{m}(B_k),$$

where the second equality is justified by Tonelli's theorem. Thus since $\mathrm{m}(\mathcal{O}^n) \to 0$ as $n \to \infty$, for any fixed $k \in \mathbb{N}$, $\mathrm{m}(\widehat{\mathcal{O}}_k^n) \to 0$, as $n \to \infty$. Then, since $\widetilde{\mathcal{Z}}_k \subset \widehat{\mathcal{O}}_k^n$, for all n we get

$$m^*(\widetilde{\mathcal{Z}}_k) = 0.$$

Then finally, since $\widetilde{\mathcal{Z}} = \bigcup_{k=1}^{\infty} \widetilde{\mathcal{Z}}_k$,

$$0 \leq \mathrm{m}^*(\widetilde{\mathcal{Z}}) \leq \sum_{k=1}^{\infty} \mathrm{m}^*(\widetilde{\mathcal{Z}}_k) = 0.$$

This proves that $\widetilde{\mathcal{Z}}$ is measurable and we deduce the result.