Serie 5

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Compute the following limits and justify your computations.

(i) $\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \arctan(x)\, dx\,.$

(ii) $\lim_{n\to\infty} \int_0^1 n^2 (1-x)^n \sin(\pi x)\, dx\,.$

(iii) $\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{1+n^2x^2}\, dx\,.$

Exercise 2. For $a \in \mathbb{R}$ consider

$$f_a(x,y) := \begin{cases} \frac{1}{(1+|x|)^a} e^{xy} & \text{if } (x,y) \in \mathbb{R} \times [x - e^{-x^2}, x]\,, \\ 0 & \text{else}\,. \end{cases}$$

Determine for which values of a it holds $f_a \in L^1(\mathbb{R}^2)$. Then compute (and justify your computation) $\lim_{a\to\infty} \int_{\mathbb{R}^2} f_a(x,y)\, dx\, dy\,.$

Exercise 3. Prove or disprove the following statement: Let $f : (a,b) \to \mathbb{R}$ be absolutely integrable such that

$$\int_a^x f(y)\, dy = 0 \quad \forall x \in (a,b).$$

Then $f = 0$ almost everywhere.

Exercise 4. We show that translations are continuous on $L^p(\mathbb{R}^n)$. In other words, let $f \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and prove that

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p\, dx = 0.$$

Hint: Begin by showing the result for $f \in C_c^\infty(\mathbb{R}^n)$ and then approximate any function in $L^p(\mathbb{R}^n)$ by functions in $C_c^\infty(\mathbb{R}^n)$ in order to conclude.

Exercise 5. We prove a kind of continuity of the Lebesgue measure under translations.

(i) Let A be a measurable set and $\mathrm{m}(A) < \infty$. Show that

$$\lim_{|\varepsilon|\to 0} \mathrm{m}\big((A+\varepsilon)\setminus A\big) = 0.$$

(ii) Show that the result in (i) is false if $\mathrm{m}(A) = \infty$.

(iii) Show that the result in (i) is false if A is not measurable. (Replace m by m^*.)

Exercise 6. The goal of this exercise is to define a notion of **dimension** for general sets $E \subseteq \mathbb{R}^d$ and to compute the dimension of the Cantor set. In order to do so, we first need to introduce the s-Haussdorf measure of a set $E \subseteq \mathbb{R}^d$, see the following definitions.

i) The diameter of E is the maximal distance between points in E, i.e.

$$diam(E) := \sup\{|x-y| : x, y \in E\}.$$

ii) For any $\delta \in (0,\infty]$, we say that $\{F_n\}_{n\in\mathbb{N}}$, with $F_n \subseteq \mathbb{R}^d$, is a δ-covering of E if

$$E \subseteq \bigcup_{n\in\mathbb{N}} F_n \qquad \text{and} \qquad diam(F_n) < \delta \quad \forall n \in \mathbb{N}.$$

iii) Let $s \in [0,\infty)$. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) := \inf\left\{\sum_{n=0}^{\infty} diam(F_n)^s : \{F_n\}_{n\in\mathbb{N}}\ \delta\text{-covering of } E\right\} \in [0,\infty]$$

Notice that $\mathcal{H}^s_\epsilon(E) \le \mathcal{H}^s_\delta(E)$, whenever $\epsilon \ge \delta$.

iv) Finally, we define the s-**Haussdorf measure** of E as

$$\mathcal{H}^s(E) := \sup_{\delta>0} \mathcal{H}^s_\delta(E) \in [0,\infty].$$

One can prove that $\mathcal{H}^s$ is in fact a measure, when restricted to the σ-algebra of Borel subsets of $\mathbb{R}^d$. Moreover, up to a multiplicative constant, $\mathcal{H}^d$ coincides with the Lebesgue measure in $\mathbb{R}^d$, and, more generally, the $\mathcal{H}^k$-measure of a k-surface in $\mathbb{R}^d$ coincides with its k-surface measure.

(i) Prove that $\mathcal{H}^0$ coincides with the "counting measure", i.e.

$$\mathcal{H}^0(E) = \begin{cases} \#E & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\#E$ stands for the cardinality of the set E.

(ii) Take $0 \leq s < t < \infty$. Prove the following implications:

$$\mathcal{H}^s(E) < \infty \implies \mathcal{H}^t(E) = 0; \qquad \mathcal{H}^t(E) > 0 \implies \mathcal{H}^s(E) = \infty.$$

Deduce that for any $E \subset \mathbb{R}^d$ there exists a unique number $s^* =: \dim_{\mathcal{H}}(E)$ such that

$$\mathcal{H}^s(E) = \infty \quad \text{for every } s \in [0, s^*), \qquad \text{and} \qquad \mathcal{H}^s(E) = 0 \quad \text{for every } s \in (s^*, \infty].$$

$\dim_{\mathcal{H}}(E)$ is called the **Haussdorf dimension** of the set E. One can prove that this notion of dimension coincides with the intuitive one for integer values: for instance, if E is a k-dimensional vector space or a k-surface in $\mathbb{R}^d$, then $\dim_{\mathcal{H}}(E) = k$. However, as we will see in the next point, not all sets need to have an integer dimension!

(iii) Let $P \subset [0, 1]$ be the Cantor set. Prove that

$$\dim_{\mathcal{H}}(P) = \frac{\log(2)}{\log(3)}.$$

Hints:

i) To show the inequality $\dim(P) \leq \log(2)/\log(3)$, remember that $P = \bigcap_{k\in\mathbb{N}} P_k$, where P_k is the union of 2^k disjoint intervals of length 3^{-k}. Then use the definition of Haussdorf measure and the characterization of the Haussdorf dimension.

ii) Observe that to prove $\dim(P) \geq \log(2)/\log(3) =: \alpha$ it is enough to show that $\mathcal{H}^\alpha(P) > 0$. To do so, you can proceed along the following steps:

(a) Show that $\mathcal{H}^s(P)$ does not change if in the definition of a δ-covering $\{F_n\}_{n\in\mathbb{N}}$ of P we require all the sets F_n to be open intervals.

(b) Take a δ-covering $\{F_n\}_{n\in\mathbb{N}}$ of P made of open intervals. Use compactness to prove that for some $N, k_0 \in \mathbb{N}$ large enough, $\{F_n\}_{n=0}^N$ is a covering of P_k, for every $k \geq k_0$.

(c) Again by compactness, prove that for some $k_1 \geq k_0$ large enough, each of the 2^{k_1} intervals of length 3^{-k_1} composing P_{k_1} is included in at least one of the open intervals $\{F_n\}_{n\in\mathbb{N}}$.

(d) By the previous step, for every $i \in \{1, \dots, 2^{k_1}\}$ there is some $n(i) \in \{0, \dots, N\}$ such that $I_i^{k_1} \subset F_{n(i)}$. Call $C_n := \{1 \leq i \leq 2^{k_1} : n(i) = n\}$. Using the definition of the Cantor set, prove the following inequality:

$$diam(F_n)^\alpha \geq \frac{1}{4} \cdot \#C_n \cdot 3^{-\alpha k_1} \qquad \forall n = 0, \dots, N.$$

Deduce from it that $\mathcal{H}^\alpha(P) \geq 1/4$.