Serie 5
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Compute the following limits and justify your computations.

(i) $\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \arctan(x)\, dx$.

(ii) $\lim_{n\to\infty} \int_0^1 n^2 (1-x)^n \sin(\pi x)\, dx$.

(iii) $\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{1+n^2x^2}\, dx$.

Solution:

(i) The monotone convergence theorem implies

$$\int_0^\infty n^2 e^{-nx} \arctan(x)dx = \lim_{m\to\infty} \int_0^\infty n^2 e^{-nx} \arctan(x)\mathbf{1}_{[0,m]}(x)dx$$
$$= \lim_{m\to\infty} \int_0^m n^2 e^{-nx} \arctan(x)dx.$$

If $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable on the compact interval $[a,b]$, then f is absolutely integrable and the Riemann integral $(R)\int f$ and the Lebesgue integral $\int f$ of f on $[a,b]$ coincide. In that case, we can use the usual change of variables formula proved for Riemann integration. So applying the change of variables $y = nx$ (since the integrand below is continuous, it is Riemann integrable),

$$\int_0^m n^2 e^{-nx} \arctan(x)dx = (R)\int_0^m n^2 e^{-nx} \arctan(x)dx = (R)\int_0^{mn} ne^{-y} \arctan\left(\frac{y}{n}\right)dy$$
$$= \int_0^{mn} ne^{-y} \arctan\left(\frac{y}{n}\right)dy = \int_0^\infty ne^{-y} \arctan\left(\frac{y}{n}\right)\mathbf{1}_{[0,mn]}(y)dy.$$

Applying the monotone convergence theorem again (for the second equality below),

$$\int_0^\infty n^2 e^{-nx} \arctan(x)dx = \lim_{m\to\infty} \int_0^\infty ne^{-y} \arctan\left(\frac{y}{n}\right)\mathbf{1}_{[0,mn]}(y)dy = \int_0^\infty ne^{-y} \arctan\left(\frac{y}{n}\right)dy$$
$$= \int_0^\infty \underbrace{ye^{-y}\frac{\arctan\left(\frac{y}{n}\right)}{y/n}}_{=:f_n(y)} dy.$$

Since $\frac{\arctan(u)}{u} \leq 1\,\forall u \geq 0$, the function f_n is dominated on $(0,\infty)$ by $f(y) = ye^{-y}$, which is Lebesgue integrable on $[0,\infty)$, and $\lim_{n\to\infty} f_n(y) = f(y)$. So using the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \arctan(x)dx = \lim_{n\to\infty} \int_0^\infty ye^{-y}\frac{\arctan\left(\frac{y}{n}\right)}{y/n}dy = \int_0^\infty ye^{-y}dy.$$

Lastly, let us compute $\int_0^\infty ye^{-y}dy$. By the monotone convergence theorem, we have

$$\int_0^\infty ye^{-y}dy = \lim_{k\to\infty} \int_0^\infty ye^{-y}\mathbf{1}_{[0,k]}dy = \lim_{k\to\infty} \int_0^k ye^{-y}dy$$
$$= \lim_{k\to\infty} (R) \int_0^k ye^{-y}dy = \lim_{k\to\infty} (1-(k+1)e^{-k}) = 1.$$

(ii) If $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, then f is absolutely integrable and the Riemann integral $(R)\int f$ and the Lebesgue integral $\int f$ of f on $[a,b]$ coincide. In that case, we can use the usual change of variables formula proved for Riemann integration. So applying the change of variables $y = nx$, we have

$$\int_0^1 n^2(1-x)^n \sin(\pi x)dx = (R)\int_0^1 n^2(1-x)^n \sin(\pi x)dx$$
$$= (R)\int_0^n n\left(1-\frac{y}{n}\right)^n \sin\left(\pi\frac{y}{n}\right)dy$$
$$= \int_0^n n\left(1-\frac{y}{n}\right)^n \sin\left(\pi\frac{y}{n}\right)dy = \int_\Omega f_n(y)dy,$$

where $\Omega = (0,\infty)$ and

$$f_n(y) = n\left(1-\frac{y}{n}\right)^n \sin\left(\pi\frac{y}{n}\right)\mathbf{1}_{[0,n]}(y) = \pi y\left(1-\frac{y}{n}\right)^n \frac{\sin\left(\pi\frac{y}{n}\right)}{\pi y/n}\mathbf{1}_{[0,n]}(y).$$

Observe that the pointwise limit of f_n is $f(y) = \pi ye^{-y}$, and each f_n is dominated by f because:

- $\sin(x) \leq x$ for all $x \geq 0$, and $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$;
- the sequence $\{(1-\frac{y}{n})^n\}_{n\geq 1}$ is increasing and has limit e^{-y}.

Hence, the dominated convergence theorem implies

$$\lim_{n\to\infty}\int_0^1 n^2(1-x)^n \sin(\pi x)dx = \lim_{n\to\infty}\int_\Omega f_n(y)dy = \int_\Omega \pi y e^{-y}dy = \pi,$$

using the computation from the previous exercise.

(iii) We can actually explicitly compute the integral and limit (again using that if $f\colon [a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then the Riemann integral and the Lebesgue integral of f on $[a,b]$ coincide):

$$\lim_{n\to\infty}\int_0^1 \frac{n^{3/2}x}{1+n^2x^2}dx = \lim_{n\to\infty}\left[\frac{1}{2\sqrt{n}}\log(1+n^2x^2)\right]_0^1 = \lim_{n\to\infty}\frac{1}{2\sqrt{n}}\log(1+n^2) = 0$$

Exercise 2. For $a\in\mathbb{R}$ consider

$$f_a(x,y) := \begin{cases} \frac{1}{(1+|x|)^a}e^{xy} & \text{if } (x,y)\in\mathbb{R}\times[x-e^{-x^2},x]\,, \\ 0 & \text{else}\,. \end{cases}$$

Determine for which values of a it holds $f_a\in L^1(\mathbb{R}^2)$. Then compute (and justify your computation) $\lim_{a\to\infty}\int_{\mathbb{R}^2} f_a(x,y)\,dx\,dy$.

Solution: By Tonelli's theorem,

$$I_a := \int_{\mathbb{R}^2}|f_a|\,d(x,y) = \int_{-\infty}^{\infty}\int_{x-e^{-x^2}}^{x}\frac{e^{xy}}{(1+|x|)^a}\,dy\,dx = \int_{-\infty}^{\infty}\frac{1}{(1+|x|)^a}\underbrace{\frac{1}{x}e^{x^2}\left(1-e^{-xe^{-x^2}}\right)}_{=:g(x)}dx.$$

To compute the integral I_a we can split it in two parts:

$$I_a = \underbrace{\int_{-\infty}^0\frac{g(x)}{(1+|x|)^a}dx}_{=:I_a^1} + \underbrace{\int_0^\infty\frac{g(x)}{(1+|x|)^a}dx}_{=:I_a^2}.$$

Observe

$$\lim_{x\to\infty}\frac{1}{x}e^{x^2}\left(1-e^{-xe^{-x^2}}\right) = 1, \qquad \lim_{x\to-\infty}\frac{1}{x}e^{x^2}\left(1-e^{-xe^{-x^2}}\right) = 1.$$

Therefore, we can upper and lower bound $\frac{g(x)}{(1+|x|)^a}$ by functions of the form $\frac{C}{(1+|x|)^a}$ (where $C>0$ is a constant). Thus, I_a^1 and I_a^2 are each finite if and only if $a>1$. Lastly, I_a is finite if and only if I_a^1 and I_a^2 are finite. Therefore, $f_a\in L^1(\mathbb{R}^2)$ if and only if $a>1$.

Now let us compute the integral $\lim_{a\to\infty}\int_{\mathbb{R}^2}f_a(x,y)\,dx\,dy = \lim_{a\to\infty}\int_{\mathbb{R}}\frac{g(x)}{(1+|x|)^a}dx$. Note that for $a\geq 2$, the integrand $\frac{g(x)}{(1+|x|)^a}$ is dominated on $\mathbb{R}$ by $\frac{2}{(1+|x|)^2}$, which is absolutely integrable on $\mathbb{R}$. Also,

$$\lim_{a\to\infty}\frac{g(x)}{(1+|x|)^a} = \left(\begin{cases}1 & \text{if } x=0\\ 0 & \text{otherwise}\end{cases}\right) = \mathbf{1}_{\{0\}}.$$

Therefore, the dominated convergence theorem implies

$$\lim_{a\to\infty}\int_{\mathbb{R}^2} f_a(x,y)\,dx\,dy \;=\lim_{a\to\infty}\int_{\mathbb{R}}\frac{g(x)}{(1+|x|)^a}dx=\int_{\mathbb{R}}\mathbf{1}_{\{0\}}(x)dx=0.$$

Exercise 3. Prove or disprove the following statement: Let $f:(a,b)\to\mathbb{R}$ be absolutely integrable such that

$$\int_a^x f(y)\,dy=0\quad \forall x\in(a,b).$$

Then $f=0$ almost everywhere.

Solution: We will prove that the statement is true. The solution consists in several steps.

Step 1: We prove that

$$\int_U f(y)\,dy=0\qquad \textit{for any open set } U\subseteq(a,b)\,.$$

First of all, notice that for any $a<c\le d<b$, we have by assumption

$$\int_c^d f(y)\,dy=\int_a^d f(y)\,dy-\int_a^c f\,dy=0. \tag{1}$$

Recall that every open set $U\subseteq(a,b)$ can be written as a countable union of pairwise disjoint, open intervals (see for instance exercise sheet 1). Define for every $N\ge 1$, the sets

$$U_N=\bigcup_{n=1}^N(c_n,d_n)\,.$$

Observe $f\mathbb{1}_{U_N}\to f\mathbb{1}_U$ pointwise everywhere and $|f\mathbb{1}_{U_N}|,|f\mathbb{1}_U|\le|f|$ pointwise. Since $|f|$ is integrable, the dominated convergence theorem gives us

$$\int_U f(y)\,dy=\int_{(a,b)}f(y)\mathbb{1}_U(y)\,dy=\lim_{N\to\infty}\int_{(a,b)}f(y)\mathbb{1}_{U_N}(y)\,dy=\lim_{N\to\infty}\sum_{n=1}^N\int_{(c_n,d_n)}f(y)\,dy=0$$

where the last equality follows from (1).

Step 2: We prove that

$$\int_A f(y)\,dy=0\qquad \textit{for any measurable set } A\subseteq(a,b)\,.$$

Let $A\subseteq(a,b)$ measurable. For any $n\in\mathbb{N}$, there is an open set U_n such that $A\subseteq U_n$ and $\mathrm{m}(U_n\setminus A)\le 1/n$. Notice that $\mathbb{1}_{U_n}\to\mathbb{1}_A$ pointwise almost everywhere as $n\to\infty$ and therefore $f\mathbb{1}_{U_n}\to f\mathbb{1}_A$ pointwise almost everywhere. Moreover, $|f\mathbb{1}_{U_n}|\le|f|$ so that by the dominated convergence theorem

$$\int_A f(y)\,dy=\lim_{n\to\infty}\int_{U_n}f(y)\,dy=0.$$

Step 3: Conclusion.

Define $A_1 := \{x \in (a,b) : f(x) > 0\}$ and $A_2 := \{x \in (a,b) : f(x) < 0\}$. From the measurability of f it follows that A_1 and A_2 are measurable. From Step 2, we deduce that

$$\int_{A_1} |f|(y)\, dy = \int_{A_1} f(y)\, dy = 0 \quad \text{and} \quad \int_{A_2} |f|(y)\, dy = -\int_{A_2} f(y)\, dy = 0\,.$$

We deduce that $|f| = 0$ almost everywhere on $A_1 \cup A_2$, hence $|f| = 0$ almost everywhere.

Exercise 4. We show that translations are continuous on $L^p(\mathbb{R}^n)$. In other words, let $f \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and prove that

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p\, dx = 0.$$

Hint: Begin by showing the result for $f \in C_c^\infty(\mathbb{R}^n)$ and then approximate any function in $L^p(\mathbb{R}^n)$ by functions in $C_c^\infty(\mathbb{R}^n)$ in order to conclude.

Solution: We split the proof in two steps.

Step 1: We prove the result in the case of $f \in C_c^\infty(\mathbb{R}^n)$.

If $f \in C_c^\infty(\mathbb{R}^n)$, there exists $R > 0$ such that $\mathrm{supp} f \subseteq B(0,R)$. For any $\varepsilon \in \mathbb{R}^d$ such that $|\varepsilon| < 1$, we define the function $G_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by $G_\varepsilon(x) = |f(x+\varepsilon) - f(x)|^p$. Note that $|G_\varepsilon(x)| \le 2^p \|f\|^p_{C^0(\mathbb{R}^n)}$ if $x \in B(0, R+1)$ and $G_\varepsilon(x) = 0$ if $x \notin B(0, R+1)$; in particular G_ε is bounded and has compact support, hence $G_\varepsilon \in L^1(\mathbb{R}^n)$. Moreover, $G_\varepsilon(x) \to 0$ pointwise as $|\varepsilon| \to 0$. Thus, by the dominated convergence theorem, we have

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p\, dx = \lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} G_\varepsilon(x)\, dx = 0.$$

Step 2: We conclude the result for a general $f \in L^p(\mathbb{R}^n)$ by approximation.

Let $\tau > 0$. By density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, for any $\tau > 0$ there exists $g \in C_c^\infty(\mathbb{R}^n)$, such that

$$||g - f||^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |g(x) - f(x)|^p\, dx \le \frac{\tau}{3^p},$$

which directly implies that for every ε we also have

$$\int_{\mathbb{R}^n} |g(x+\varepsilon) - f(x+\varepsilon)|^p\, dx \le \frac{\tau}{3^p}\,.$$

Since $g \in C_c^\infty(\mathbb{R}^n)$ we know from Step 1 that

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |g(x+\varepsilon) - g(x)|^p\, dx = 0,$$

and therefore for all $\varepsilon = \varepsilon(\tau)$ small enough, we have

$$\int_{\mathbb{R}^n} |g(x+\varepsilon) - g(x)|^p\, dx \le \frac{\tau}{3^p}.$$

We conclude by the triangular inequality that

$$\int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p \, dx = \int_{\mathbb{R}^n} |f(x+\varepsilon) - g(x+\varepsilon) + g(x+\varepsilon) - g(x) + g(x) - f(x)|^p \, dx$$
$$\leq \int_{\mathbb{R}^n} 3^{p-1} \left\{ |f(x+\varepsilon) - g(x+\varepsilon)|^p + |g(x+\varepsilon) - g(x)|^p + |g(x) - f(x)|^p \right\} dx$$
$$= 3^{p-1} \left\{ \int_{\mathbb{R}^n} |f(x+\varepsilon) - g(x+\varepsilon)|^p \, dx + \int_{\mathbb{R}^n} |g(x+\varepsilon) - g(x)|^p \, dx + \int_{\mathbb{R}^n} |g(x) - f(x)|^p \, dx \right\} \leq \tau.$$

The previous inequality being true for any τ, we deduce the result letting $\tau \to 0$.

Exercise 5. We prove a kind of continuity of the Lebesgue measure under translations.

(i) Let A be a measurable set and $\mathrm{m}(A) < \infty$. Show that

$$\lim_{|\varepsilon| \to 0} \mathrm{m}\big((A+\varepsilon) \setminus A\big) = 0.$$

(ii) Show that the result in (i) is false if $\mathrm{m}(A) = \infty$.

(iii) Show that the result in (i) is false if A is not measurable. (Replace m by m^*.)

Solution:

(i) Let $f = \chi_A$. Since $f \in L^1(\mathbb{R}^n)$, using exercise 5, we get

$$\lim_{|\varepsilon| \to 0} \int |\chi_A(x-\varepsilon) - \chi_A(x)| dx = 0.$$

Since

$$|\chi_A(x-\varepsilon) - \chi_A(x)| = \chi_{[(A+\varepsilon)\setminus A] \cup [A \setminus (A+\varepsilon)]}(x),$$

it follows that

$$\int |\chi_A(x-\varepsilon) - \chi_A(x)| dx = \mathrm{m}\big([(A+\varepsilon) \setminus A] \cup [A \setminus (A+\varepsilon)]\big).$$

Finally, using

$$\mathrm{m}((A+\varepsilon) \setminus A) \leq \mathrm{m}([(A+\varepsilon) \setminus A] \cup [A \setminus (A+\varepsilon)])$$

we deduce the result.

(ii) Define

$$A := \bigcup_{n=1}^{\infty} (n, n+1/2).$$

Then for all $0 < \varepsilon < 1/2$,

$$(A+\varepsilon) \setminus A = \bigcup_{n=1}^{\infty} [n+1/2, n+1/2+\varepsilon),$$

and by additivity,

$$\mathrm{m}\left((A+\varepsilon)\setminus A\right) = \sum_{n=1}^{\infty} \varepsilon = \infty.$$

(iii) Let $V \subset [0,1]$ be the non-measurable Vitali. Recall from the construction that V has the following property:

$$(V+\varepsilon)\cap V = \emptyset \quad \forall \varepsilon \in \mathbb{Q}\cap[-1,1]\setminus\{0\}.$$

It follows from translation invariance of the outer measure that for any $\varepsilon \in \mathbb{Q}\cap[-1,1]\setminus\{0\}$, it holds

$$\mathrm{m}^*\left((V+\varepsilon)\setminus V\right) = \mathrm{m}^*(V+\varepsilon) = \mathrm{m}^*(V).$$

Notice that necessarily $\mathrm{m}^*(V) > 0$ (otherwise V would be measurable), hence

$$\lim_{\substack{|\varepsilon|\to 0 \\ \varepsilon\in\mathbb{Q}\cap[-1,1]\setminus\{0\}}} \mathrm{m}^*((V+\varepsilon)\setminus V) = \mathrm{m}^*(V) > 0\,.$$

Exercise 6. The goal of this exercise is to define a notion of **dimension** for general sets $E \subseteq \mathbb{R}^d$ and to compute the dimension of the Cantor set. In order to do so, we first need to introduce the s-Haussdorf measure of a set $E \subseteq \mathbb{R}^d$, see the following definitions.

i) The diameter of E is the maximal distance between points in E, i.e.

$$diam(E) := \sup\left\{|x-y| : x, y \in E\right\}.$$

ii) For any $\delta \in (0,\infty]$, we say that $\{F_n\}_{n\in\mathbb{N}}$, with $F_n \subseteq \mathbb{R}^d$, is a δ-covering of E if

$$E \subseteq \bigcup_{n\in\mathbb{N}} F_n \qquad \text{and} \qquad diam(F_n) < \delta \quad \forall n \in \mathbb{N}.$$

iii) Let $s \in [0,\infty)$. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) := \inf\left\{\sum_{n=0}^{\infty} diam(F_n)^s : \{F_n\}_{n\in\mathbb{N}}\ \delta\text{-covering of } E\right\} \in [0,\infty]$$

Notice that $\mathcal{H}^s_\epsilon(E) \leq \mathcal{H}^s_\delta(E)$, whenever $\epsilon \geq \delta$.

iv) Finally, we define the **s-Haussdorf measure** of E as

$$\mathcal{H}^s(E) := \sup_{\delta>0} \mathcal{H}^s_\delta(E) \in [0,\infty].$$

One can prove that $\mathcal{H}^s$ is in fact a measure, when restricted to the σ-algebra of Borel subsets of $\mathbb{R}^d$. Moreover, up to a multiplicative constant, $\mathcal{H}^d$ coincides with the Lebesgue measure in $\mathbb{R}^d$, and, more generally, the $\mathcal{H}^k$-measure of a k-surface in $\mathbb{R}^d$ coincides with its k-surface measure.

(i) Prove that $\mathcal{H}^0$ coincides with the "counting measure", i.e.

$$\mathcal{H}^0(E) = \begin{cases} \#E & \text{if } E \text{ is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\#E$ stands for the cardinality of the set E.

(ii) Take $0 \leq s < t < \infty$. Prove the following implications:

$$\mathcal{H}^s(E) < \infty \implies \mathcal{H}^t(E) = 0; \qquad \mathcal{H}^t(E) > 0 \implies \mathcal{H}^s(E) = \infty.$$

Deduce that for any $E \subset \mathbb{R}^d$ there exists a unique number $s^* =: \dim_{\mathcal{H}}(E)$ such that

$$\mathcal{H}^s(E) = \infty \quad \text{for every } s \in [0, s^*), \qquad \text{and} \qquad \mathcal{H}^s(E) = 0 \quad \text{for every } s \in (s^*, \infty].$$

$\dim_{\mathcal{H}}(E)$ is called the **Haussdorf dimension** of the set E. One can prove that this notion of dimension coincides with the intuitive one for integer values: for instance, if E is a k-dimensional vector space or a k-surface in $\mathbb{R}^d$, then $\dim_{\mathcal{H}}(E) = k$. However, as we will see in the next point, not all sets need to have an integer dimension!

(iii) Let $P \subset [0, 1]$ be the Cantor set. Prove that

$$\dim_{\mathcal{H}}(P) = \frac{\log(2)}{\log(3)}.$$

Hints:

i) To show the inequality $\dim(P) \leq \log(2)/\log(3)$, remember that $P = \bigcap_{k \in \mathbb{N}} P_k$, where P_k is the union of 2^k disjoint intervals of length 3^{-k}. Then use the definition of Haussdorf measure and the characterization of the Haussdorf dimension.

ii) Observe that to prove $\dim(P) \geq \log(2)/\log(3) =: \alpha$ it is enough to show that $\mathcal{H}^\alpha(P) > 0$. To do so, you can proceed along the following steps:

(a) Show that $\mathcal{H}^s(P)$ does not change if in the definition of a δ-covering $\{F_n\}_{n \in \mathbb{N}}$ of P we require all the sets F_n to be open intervals.

(b) Take a δ-covering $\{F_n\}_{n \in \mathbb{N}}$ of P made of open intervals. Use compactness to prove that for some $N, k_0 \in \mathbb{N}$ large enough, $\{F_n\}_{n=0}^N$ is a covering of P_k, for every $k \geq k_0$.

(c) Again by compactness, prove that for some $k_1 \geq k_0$ large enough, each of the 2^{k_1} intervals of length 3^{-k_1} composing P_{k_1} is included in at least one of the open intervals $\{F_n\}_{n \in \mathbb{N}}$.

(d) By the previous step, for every $i \in \{1, \dots, 2^{k_1}\}$ there is some $n(i) \in \{0, \dots, N\}$ such that $I_i^{k_1} \subset F_{n(i)}$. Call $C_n := \{1 \leq i \leq 2^{k_1} : n(i) = n\}$. Using the definition of the Cantor set, prove the following inequality:

$$diam(F_n)^\alpha \geq \frac{1}{4} \cdot \#C_n \cdot 3^{-\alpha k_1} \qquad \forall n = 0, \dots, N.$$

Deduce from it that $\mathcal{H}^\alpha(P) \geq 1/4$.

Solution:

(i) Assume first that $E = \{p_1, \dots, p_N\}$ is finite. We want to show that $\mathcal{H}^0(E) = N$. Call $\rho > 0$ the minimal separation between two points of E, i.e.

$$\rho := \min\{|p_i - p_j| : 1 \le i < j \le N\}.$$

Let $\delta < \rho/2$ and $\{F_n\}_{n\in\mathbb{N}}$ be a δ-covering of E. Then, since each set F_n has diameter less than $\delta < \rho/2$ and any two points in E are separated at least by ρ, we deduce that there are at least N non-empty sets in the covering $\{F_n\}_{n\in\mathbb{N}}$. Therefore

$$\sum_{n\in\mathbb{N}} diam(F_n)^0 \ge N.$$

By arbitrariness of the δ-covering $\{F_n\}_{n\in\mathbb{N}}$ we deduce $\mathcal{H}^0_\delta(E) \ge N$, and so, in particular $\mathcal{H}^0(E) \ge 0$. For the opposite inequality, we just notice that, for any $\delta > 0$, $\{B_{\delta/2}(p_i)\}_{i=1}^N$ is a δ-covering of E and

$$\mathcal{H}^0_\delta(E) \le \sum_{i=1}^N diam(B_{\delta/2}(p_i))^0 = N,$$

therefore $\mathcal{H}^0(E) = \sup_{\delta>0} \mathcal{H}^0_\delta(E) \le N$. Finally, to prove that $\mathcal{H}^0(E) = +\infty$ for E infinite we just observe that Haussdorf measures are clearly monotone non-decreasing with respect to inclusion.

(ii) We prove only the first implication, as the second is analogous. Let $E \subset \mathbb{R}^d$ be such that $\mathcal{H}^s(E) = L < \infty$. Then, by the definition of Haussdorf measure, for every $k \ge 1$ there exists a $(1/k)$-covering $\{F_n^k\}_{n\in\mathbb{N}}$ of E such that

$$\sum_{n\in\mathbb{N}} diam(F_n^k)^s \le L + 1.$$

Hence,

$$\mathcal{H}^t_{1/k}(E) \le \sum_{n\in\mathbb{N}} diam(F_n^k)^t \le \frac{1}{k^{t-s}} \sum_{n\in\mathbb{N}} diam(F_n^k)^s \le \frac{L+1}{k^{t-s}}.$$

In particular, for a fixed $\delta > 0$, we have

$$\mathcal{H}^t_\delta(E) \le \mathcal{H}^t_{1/k}(E) \le \frac{L+1}{k^{t-s}} \quad \text{for any } k \in \mathbb{N} \text{ such that } 1/k \le \delta,$$

that is, $\mathcal{H}^t_\delta(E) = 0$ for every $\delta > 0$, and so $\mathcal{H}^t(E) = 0$.

(iii) Let $\alpha := \log(2)/\log(3)$. We prove separately the two inequalities $\dim_{\mathcal{H}}(P) \le \alpha$ and $\dim_{\mathcal{H}}(P) \ge \alpha$. Recall that $P = \bigcap_{k\in\mathbb{N}} P_k$, where $P_{k+1} \subset P_k$, and $P_k = \bigcup_{i=1}^{2^k} I_i^k$ is the union of 2^k disjoint closed intervals $\{I_i^k\}_{i=1}^{2^k}$ each one of length (and hence diameter) 3^{-k}.

- ($\dim_{\mathcal{H}}(P) \le \alpha$). By the characterization of the Haussdorf dimension it is enough to show that $\mathcal{H}^\alpha(P) < \infty$. Take $\delta > 0$, and choose $k \in \mathbb{N}$ large enough such that $3^{-k} \le \delta$.

Then $\{I_i^k\}_{i=1}^{2^k}$ is a δ-covering of P. Therefore

$$\mathcal{H}^\alpha_\delta(P) \leq \sum_{i=1}^{2^k} diam(I_i^k)^\alpha = 2^k 3^{-k\alpha} = 1,$$

where in the last equality we have used the definition of α. Therefore, $\mathcal{H}^\alpha(P) \leq 1 < \infty$ and the first inequality is proved.

- ($\dim_{\mathcal{H}}(P) \geq \alpha$). Again by the characterization of the Haussdorf dimension it is enough to show that $\mathcal{H}^\alpha(P) > 0$. We proceed with the strategy outlined in the hints.

  a) Here we show that $\mathcal{H}^\alpha_\delta(P)$ remains unchainged if in the definition of δ-covering we require the sets $\{F_n\}_{n\in\mathbb{N}}$ to be open intervals. To do so, we just need to show that for any δ-covering $\{F_n\}_{n\in\mathbb{N}}$ of P and any $\epsilon > 0$ we may find a δ-covering $\{G_n\}_{n\in\mathbb{N}}$ of P such that G_n are open intervals and

  $$\sum_{n\in\mathbb{N}} diam(G_n)^\alpha \leq (1+\epsilon)\sum_{n\in\mathbb{N}} diam(F_n)^\alpha.$$

  We may assume without loss of generality that $diam(F_n) < \infty$ for every $n \in \mathbb{N}$. Then, the etremals $a_n := \inf F_n$, and $b_n := \sup F_n$ of F_n are well-defined, $-\infty < a_n \leq b_n \leq +\infty$ and $diam F_n = b_n - a_n$. Then we may take $\epsilon_n > 0$ sufficiently small so that $b_n - a_n + 2\epsilon_n < (b_n - a_n)(1+\epsilon)^{1/\alpha}$ and define

  $$G_n := (a_n - \epsilon_n, b_n + \epsilon_n),$$

  thus concluding this step.

  b) Now we consider any covering $\{F_n\}_{n\in\mathbb{N}}$ of P made of open intervals. Our final goal is to give a lower bound for $\sum_{n\in\mathbb{N}} diam(F_n)^\alpha$ independent of the covering. Observe that since P is compact, we may extract from $\{F_n\}_{n\in\mathbb{N}}$ a finite sub-covering $\{F_n\}_{n=0}^N$ of P. Let us show that for some $k_0 \in \mathbb{N}$ large enough, $\{F_n\}_{n=0}^N$ covers P_{k_0}, (and thus it also covers P_k for every $k \geq k_0$). If, by contradiction, this was not the case, we would have a sequence of points $x_k \in P_k \setminus \bigcup_{n=0}^N F_n$. Then up to subsequences, $x_k \to x_\infty \in P \setminus \bigcup_{n=0}^N F_n$, a contradiction, since $\{F_n\}_{n=0}^N$ covers P.

  c) In this step we want to show that if $k_1 \geq k_0$ is large enough, then each interval $I_i^{k_1}$ composing P_{k_1} is included in at least one of the open intervals F_n, for $n = 0, \ldots, N$. The proof is again by contradiction using compactness. Suppose we have, for every $k \geq k_0$, an interval $I_{i_k}^k = (a_k, b_k)$ with $b_k - a_k = 3^{-k}$ such that $(a_k, b_k) \notin F_n$, for any $n = 0, \ldots, N$. Then up to extracting a subsequence, there exist a point $p \in P$ such that $a_k, b_k \to p$. However, since $P \subset \bigcup_{n=0}^N F_n$, there must be some $n \in \{0, \ldots, N\}$ such that $p \in F_n$. Moreover, being F_n open, for k large enough we have $[a_k, b_k] \subset F_n$, a contradiction.

  d) In this final step we prove the desired lower bound on $\sum_{n\in\mathbb{N}} diam(F_n)^\alpha$. Let $k_1 \geq k_0$ be as in the previous step. Then, for every $i \in \{1, \ldots, 2^{k_1}\}$ there exist $n(i) \in \{0, \ldots, N\}$ such that $I_i^{k_1} \subset F_n$. We may then split $\{1, \ldots, 2^{k_1}\} = \bigcup_{n=0}^N C_n$, where $C_n := \{i : n(i) = n\}$.

We claim that

$$diam(F_n)^\alpha \geq \frac{1}{4} \cdot \#C_n \cdot 3^{-\alpha k_1} \qquad \forall n = 0, \dots, N. \tag{2}$$

Assuming the claim is proved, then we conclude as follows:

$$\sum_{n \in \mathbb{N}} diam(F_n)^\alpha \geq \sum_{n=0}^{N} diam(F_n)^\alpha \geq \frac{1}{4} 3^{-k_1 \alpha} \sum_{n=0}^{N} \#C_n = \frac{1}{4} 3^{-k_1 \alpha} 2^{k_1} = \frac{1}{4},$$

where in the last equality we used the definition of α. By the arbitrariness of the covering $\{F_n\}_{n \in \mathbb{N}}$ we deduce that $\mathcal{H}^\alpha(P) \geq 1/4$, thus concluding the proof.
Let us prove claim (**??**) now. Let $k \leq k_1$ be the smallest integer for which F_n contains some interval I_i^k. From the construction of the Cantor set, we see that F_n will intersect at most 4 intervals of the k-th generation (otherwise F_n would contain some I_i^{k-1}, thus contradicting the definition of k). Let $\mathcal{F} \subset \{I_i^k\}_{i=1}^{2^k}$ be the family of such intervals ($\#\mathcal{F} \leq 4$). We will have:

$$\begin{aligned} 4\, diam(F_n)^\alpha &\geq \sum_{I_i^k \in \mathcal{F}} diam(I_i^k)^\alpha = \sum_{I_i^k \in \mathcal{F}} 3^{-k\alpha} \\ &= \sum_{I_i^k \in \mathcal{F}} 2^{k_1 - k} 3^{-k_1 \alpha} = \sum_{I_i^k \in \mathcal{F}} \sum_{I_j^{k_1} \subset I_i^k} 3^{-k_1 \alpha} \\ &\geq \sum_{I_j^{k_1} \subset F_n} 3^{-k_1 \alpha} \geq \#C_n \cdot 3^{-k_1 \alpha}. \end{aligned}$$

In the first passage we used that $\#\mathcal{F} \leq 4$; in the third we exploited the definition of α; finally, in the fifth we used the fact that any $I_j^{k_1}$ which is included in F_n, must also be included in some I_i^k which intersect F_n.