Serie 4
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1 (Properties of signed integrals)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a measureable set and let $f, g\colon \Omega \to \mathbb{R}$ be absolutely integrable functions. Show the following statements:

(i) Prove that

$$\left|\int_\Omega f\,dx\right| \le \int_\Omega f^+\,dx + \int_\Omega f^-\,dx = \int_\Omega |f|\,dx.$$

(ii) For any real number c (positive, zero, or negative), we have that cf is absolutely integrable and

$$\int_\Omega (cf)\,dx = c\int_\Omega f\,dx\,.$$

(iii) The function $f+g$ is absolutely integrable and

$$\int_\Omega (f+g)\,dx = \int_\Omega f\,dx + \int_\Omega g\,dx\,.$$

(iv) If $f(x) \le g(x)$ for all $x \in \Omega$, then we have

$$\int_\Omega f\,dx \le \int_\Omega g\,dx.$$

(v) If $f(x) = g(x)$ for almost every $x \in \Omega$, then

$$\int_\Omega f\,dx = \int_\Omega g\,dx.$$

Hint: For (iii), break f, g and $f+g$ up into positive and negative parts, and try to write everything in terms of integrals of non-negative functions. Then, use the linearity of the integral with respect to non-negative functions.

Exercise 2. Recall from Serie 3 that if φ is measurable and f is continuous, then $f \circ \varphi$ is measurable. In general however, the composition of measurable functions is not measurable.

To see this, we define the function of Lebesgue. For $x \in [0,1]$, we consider its binary expansion

$$x := \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

with $a_n \in \{0,1\}$. As in Exercise 6 of Serie 3, this binary expansion is unique, if we identify the expansions

$$0.a_1 \cdots a_{k-1}01\cdots 1 \dots \quad \text{and } 0.a_1 \cdots a_{k-1}10\cdots 0 \dots . \tag{3}$$

We will in the sequel always assume that the expansions are of the first form, i.e. that all but finitely many a_n are equal to 1 (except for $x=0$, where $a_n = 0$ for all $n \geq 1$). With this convention, we then define $f : [0,1] \to [0,1]$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{2a_n}{3^n} . \tag{4}$$

(i) Prove that f is strictly increasing, measurable and $f([0,1]) \subseteq P$, where $P \subset [0,1]$ is the Cantor set.

(ii) Let $V \subset [0,1]$ be a non-measurable set (which you can assume to exist, see hint of Exercise 4) and define $B := f(V)$. Show that both $\mathbb{1}_B$ and f are measurable and yet, that their composition $\mathbb{1}_B \circ f$ is not measurable.

Remark: The Lebesgue function would not be well-defined without the identification (3) (Why?). Moreover, it is not true that $f([0,1]) = P$: Indeed, $\frac{2}{3} \in P$ has the tenary expansions $0.20\cdots 0\dots$ and $0.12\cdots 2\dots$. For the second expansion, we do not have a preimage, and for the first expansion, we would have the preimage $0.10\cdots 0\dots$; however *with our convention*, this is not the binary expansion of a number on $[0,1]$.

Exercise 3. In this exercise we will construct a famous example of a continuous function, the Cantor function, whose range is $[0,1]$ despite being constant almost everywhere. Recall the notation introduced in the construction of the Cantor set $P = \bigcap_{n\geq 1} P_n$ in the Lecture Notes. We define recursively a sequence $\{f\}_{n\in\mathbb{N}_{\geq 0}}$ of functions on $[0,1]$ by

$$f_0(x) = x \quad x \in [0,1],$$

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & 0 \leq x < 1/3, \\ \frac{1}{2} & 1/3 \leq x < 2/3, \\ \frac{1}{2} f_n(3x-2) + \frac{1}{2} & 2/3 \leq x \leq 1. \end{cases}$$

(i) Draw the graph of f_1, f_2, f_3 and f_4. Prove by induction that each f_n is continuous on $[0,1]$ with $f_n(0) = 0$ and $f_n(1) = 1$, monotonically increasing and constant on $[0,1] \setminus P_n$.

(ii) Prove that

$$|f_{n+1}(x) - f_n(x)| < 2^{-n} \quad \forall x \in [0,1].$$

Deduce that f_n converges uniformly on $[0,1]$ to a limit f which is continuous. We call f the Cantor function.

(iii) Prove that f is monotonically increasing on $[0,1]$ with $f(0) = 0$, $f(1) = 1$ and that f is piecewise constant on $[0,1] \setminus P$.

(iv) Deduce that f induces a bijection between P and $[0,1]$. In particular, the Cantor set P, despite being a Lebesgue null set, has the cardinality of the continuum.

Exercise 4. Show that there exists $f\colon [0,1] \to [0,1]$ continuous and two subsets $A, B \subseteq [0,1]$ such that

(i) A is measurable and $f(A)$ is not,

(ii) B is a null set and $f(B)$ has positive Lebesgue measure.

Hint: You can assume (without proof) that there exists a non-measurable subset $V \subset [0,1]$. An example of such a subset will be constructed explicitly in the lecture.

Exercise 5. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$. Prove the following statements:

(i) There exists a positive continuous function $f : \mathbb{R} \to [0, +\infty)$ which is absolutely integrable and yet $\limsup_{x\to\infty} f(x) = \infty$.

(ii) If f is absolutely integrable and $\lim_{|x|\to\infty} f(x)$ exists, then necessarily $\lim_{|x|\to\infty} f(x) = 0$.

(iii) Show that if f is uniformly continuous and absolutely integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Exercise 6 ($\star$)**.** Let $0 < \varepsilon < 1$.

(i) Construct an open dense set $E \subseteq [0,1]$ such that $\mathrm{m}(E) = \varepsilon$.

(ii) Construct a closed set F that does not contain any non-empty open set with $\mathrm{m}(F) = \varepsilon$.