Serie 4

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1 (Properties of signed integrals)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a measureable set and let $f, g \colon \Omega \to \mathbb{R}$ be absolutely integrable functions. Show the following statements:

(i) Prove that

$$\left| \int_\Omega f\,dx \right| \le \int_\Omega f^+\,dx + \int_\Omega f^-\,dx = \int_\Omega |f|\,dx.$$

(ii) For any real number c (positive, zero, or negative), we have that cf is absolutely integrable and

$$\int_\Omega (cf)\,dx = c \int_\Omega f\,dx\,.$$

(iii) The function $f + g$ is absolutely integrable and

$$\int_\Omega (f+g)\,dx = \int_\Omega f\,dx + \int_\Omega g\,dx\,.$$

(iv) If $f(x) \le g(x)$ for all $x \in \Omega$, then we have

$$\int_\Omega f\,dx \le \int_\Omega g\,dx.$$

(v) If $f(x) = g(x)$ for almost every $x \in \Omega$, then

$$\int_\Omega f\,dx = \int_\Omega g\,dx.$$

Hint: For (iii), break f, g and $f+g$ up into positive and negative parts, and try to write everything in terms of integrals of non-negative functions. Then, use the linearity of the integral with respect to non-negative functions.

Solution:

(i) Recall the triangular inequality $|x-y| \leq |x|+|y|$ for all $x, y \in \mathbb{R}$. Using that f^+ and f^- are nonnegative, we can estimate

$$\left|\int_\Omega f\,dx\right| = \left|\int_\Omega f^+\,dx - \int_\Omega f^-\,dx\right| \leq \left|\int_\Omega f^+\,dx\right| + \left|\int_\Omega f^-\,dx\right| = \int_\Omega f^+\,dx + \int_\Omega f^-\,dx$$
$$= \int_\Omega |f|\,dx.$$

(ii) First of all, notice that since f is absolutely integrable, cf is also absolutely integrable. Now, if $c=0$, the result is obvious. If c is positive, we have using the linearity of the integral for nonnegative functions that

$$\int_\Omega (cf)\,dx = \int_\Omega (cf)^+\,dx - \int_\Omega (cf)^-\,dx = \int_\Omega cf^+\,dx - \int_\Omega cf^-\,dx = c\int_\Omega f\,dx. \tag{1}$$

If c is negative instead, we have

$$\int_\Omega (cf)\,dx = \int_\Omega (cf)^+\,dx - \int_\Omega (cf)^-\,dx = \int_\Omega |c|f^-\,dx - \int_\Omega |c|f^+\,dx = -\left(\int_\Omega |c|f^+\,dx - \int_\Omega |c|f^-\,dx\right)$$
$$= -\int_\Omega |c|f\,dx \overset{(1)}{=} -|c|\int_\Omega f\,dx = c\int_\Omega f\,dx.$$

(iii) We begin by showing that $f+g$ is absolutely integrable. Indeed, by the triangular inequality and the monotonicity of the integral for nonnegative functions, we have

$$\int_\Omega |f+g|\,dx \leq \int_\Omega |f|\,dx + \int_\Omega |g|\,dx < +\infty.$$

Note that

$$(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-,$$

and so

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Therefore, using linearity of the integral for nonnegative functions,

$$\int_\Omega (f+g)^+dx + \int_\Omega f^-dx + \int_\Omega g^-dx = \int_\Omega (f+g)^-dx + \int_\Omega f^+dx + \int_\Omega g^+dx. \tag{2}$$

Thus,

$$\int_\Omega (f+g)\,dx \overset{\text{def}}{=} \int_\Omega (f+g)^+\,dx - \int_\Omega (f+g)^-\,dx$$
$$\overset{(2)}{=} \int_\Omega f^+\,dx - \int_\Omega f^-\,dx + \int_\Omega g^+\,dx - \int_\Omega g^-\,dx$$
$$\overset{\text{def}}{=} \int_\Omega f\,dx + \int_\Omega g\,dx.$$

(iv) The assumption $|f(x)| \leq |g(x)|$ guarantees $f^+(x) \leq g^+(x)$ and $g^-(x) \leq f^-(x)$ for all $x \in \Omega$ and therefore by the monotonicity of the integral for nonnegative functions

$$\int_\Omega f\,dx = \int_\Omega f^+\,dx - \int_\Omega f^-\,dx \leq \int_\Omega g^+\,dx - \int_\Omega g^-\,dx = \int_\Omega g\,dx.$$

(v) Since $f(x) = g(x)$ for almost every $x \in \Omega$, there is a set $A \subset \Omega$ of measure 0 such that $f(x)\mathbb{1}_{\Omega\setminus A}(x) = g(x)\mathbb{1}_{\Omega\setminus A}(x)$ for every $x \in \Omega$. Applying (iv) in both directions we get

$$\int_\Omega f\mathbb{1}_{\Omega\setminus A} = \int_\Omega g\mathbb{1}_{\Omega\setminus A}.$$

Thus, using (iii)

$$\int_\Omega f\,dx = \int_\Omega f\mathbb{1}_{\Omega\setminus A}\,dx + \underbrace{\int_\Omega f\,dx\mathbb{1}_A}_{=0} = \int_\Omega g\mathbb{1}_{\Omega\setminus A}\,dx + \underbrace{\int_\Omega g\mathbb{1}_A\,dx}_{=0} = \int_\Omega g\,dx\,.$$

Exercise 2. Recall from Serie 3 that if φ is measurable and f is continuous, then $f \circ \varphi$ is measurable. In general however, the composition of measurable functions is not measurable.

To see this, we define the function of Lebesgue. For $x \in [0,1]$, we consider its binary expansion

$$x := \sum_{n=1}^\infty \frac{a_n}{2^n}$$

with $a_n \in \{0,1\}$. As in Exercise 6 of Serie 3, this binary expansion is unique, if we identify the expansions

$$0.a_1 \cdots a_{k-1}01\cdots 1\ldots \qquad \text{and } 0.a_1\cdots a_{k-1}10\cdots 0\ldots\,. \tag{3}$$

We will in the sequel always assume that the expansions are of the first form, i.e. that all but finitely many a_n are equal to 1 (except for $x = 0$, where $a_n = 0$ for all $n \geq 1$). With this convention, we then define $f : [0,1] \to [0,1]$ by

$$f(x) := \sum_{n=1}^\infty \frac{2a_n}{3^n}\,. \tag{4}$$

(i) Prove that f is strictly increasing, measurable and $f([0,1]) \subseteq P$, where $P \subset [0,1]$ is the Cantor set.

(ii) Let $V \subset [0,1]$ be a non-measurable set (which you can assume to exist, see hint of Exercise 4) and define $B := f(V)$. Show that both $\mathbb{1}_B$ and f are measurable and yet, that their composition $\mathbb{1}_B \circ f$ is not measurable.

Remark: The Lebesgue function would not be well-defined without the identification (3) (Why?). Moreover, it is not true that $f([0,1]) = P$: Indeed, $\frac{2}{3} \in P$ has the tenary expansions $0.20 \cdots 0 \ldots$ and $0.12 \cdots 2 \ldots$. For the second expansion, we do not have a preimage, and for the first expansion, we would have the preimage $0.10 \cdots 0 \ldots$; however *with our convention*, this is not the binary expansion of a number on $[0,1]$.

Solution:

(i) It follows from the definition of the Cantor set, that $f([0,1]) \subset P$. It remains to prove that f is strictly increasing which implies measurability. Take $0 < x < y < 1$ and consider their binary expansions (they are unique with our convention)

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \quad \text{and} \quad y = \sum_{n=1}^{\infty} \frac{b_n}{2^n} .$$

By the convention we chose for the binary expansion, we have that if $x < y$ there exists $1 \leq k < \infty$ such that

$$a_j = b_j \text{ for all } j = 1, \ldots, k-1 \quad \text{and} \quad a_k < b_k ,$$

i.e. $a_k = 0$ and $b_k = 1$. We then have

$$\begin{aligned} f(y) - f(x) &= \frac{2}{3^k} + \sum_{n=k+1}^{\infty} \frac{2(b_n - a_n)}{3^n} \\ &\geq \frac{2}{3^k} - 2 \sum_{n=k+1}^{\infty} \frac{1}{3^n} \\ &\geq \frac{2}{3^k} - \frac{2}{3^{k+1}} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{3^k} > 0 . \end{aligned}$$

(ii) We know from (i) that f is measurable since it is strictly increasing (recall Serie 3). By construction, $B = f(V)$ is a subset of the Cantor set, so in particular $\mathrm{m}^*(B) = 0$ and hence B is Lebesgue measurable, in particular $\mathbb{1}_B$ is a measurable map. However, we have that $\mathbb{1}_B \circ f : [0,1] \to [0,1]$ and

$$\begin{aligned}\{x \in \mathbb{R} \mid (\mathbb{1}_B \circ f)(x) > 0\} =& (\mathbb{1}_B \circ f)^{-1} (]0, +\infty[) \\ =& f^{-1} \left(\mathbb{1}_B^{-1} (]0, +\infty[)\right) \\ =& f^{-1}(B) \\ =& f^{-1} (f(V)) \\ =& V \notin \mathcal{M},\end{aligned}$$

where the last equality follows from the injectivity of f. Thus, $\mathbb{1}_B \circ f$ is not measurable.

Exercise 3. In this exercise we will construct a famous example of a continuous function, the Cantor function, whose range is $[0,1]$ despite being constant almost everywhere. Recall the notation introduced in the construction of the Cantor set $P = \bigcap_{n\geq 1} P_n$ in the Lecture Notes. We define recursively a sequence $\{f\}_{n\in\mathbb{N}_{\geq 0}}$ of functions on $[0,1]$ by

$$f_0(x) = x \quad x \in [0,1],$$

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & 0 \leq x < 1/3, \\ \frac{1}{2} & 1/3 \leq x < 2/3, \\ \frac{1}{2} f_n(3x-2) + \frac{1}{2} & 2/3 \leq x \leq 1. \end{cases}$$

(i) Draw the graph of f_1, f_2, f_3 and f_4. Prove by induction that each f_n is continuous on $[0,1]$ with $f_n(0) = 0$ and $f_n(1) = 1$, monotonically increasing and constant on $[0,1] \setminus P_n$.

(ii) Prove that

$$|f_{n+1}(x) - f_n(x)| < 2^{-n} \quad \forall x \in [0,1].$$

Deduce that f_n converges uniformly on $[0,1]$ to a limit f which is continuous. We call f the Cantor function.

(iii) Prove that f is monotonically increasing on $[0,1]$ with $f(0) = 0$, $f(1) = 1$ and that f is piecewise constant on $[0,1] \setminus P$.

(iv) Deduce that f induces a bijection between P and $[0,1]$. In particular, the Cantor set P, despite being a Lebesgue null set, has the cardinality of the continuum.

Solution:

(i) It is clear that f_0 is continuous with $f_0(0) = 0$ and $f_0(1) = 1$, monotonically increasing and constant on $[0,1] \setminus P_0 = \emptyset$. We prove that the same holds for any f_n by induction.

Assume that, f_n is continuous with $f_n(0) = 0$ and $f_n(1) = 1$, monotonically increasing, constant on $[0,1] \setminus P_n$ and we want to prove the same properties for f_{n+1}. First for the continuity, it is clear from the definition that f_{n+1} is continuous on $(1/3, 2/3)$. Similarly, since f_n is continuous and the functions $x \mapsto 3x$ and $x \mapsto 3x - 2$ are continuous, f_{n+1} is continuous on $(0, 1/3)$ and $(2/3, 1)$. To get the continuity on the whole interval $[0,1]$, it suffices to notice that since $f_n(0) = 0$ and $f_n(1) = 1$, we have

$$\lim_{\substack{x \to 1/3 \\ <}} f_{n+1}(x) = \lim_{\substack{x \to 1 \\ <}} \frac{1}{2} f_n(x) = \frac{1}{2} \quad \text{and} \quad \lim_{\substack{x \to 2/3 \\ >}} f_{n+1}(x) = \lim_{\substack{x \to 0 \\ >}} \frac{1}{2} f_n(x) + \frac{1}{2} = \frac{1}{2}. \tag{5}$$

Moreover, $f_{n+1}(0) = \frac{1}{2} f_n(0) = 0$ and $f_{n+1}(1) = \frac{1}{2} f_n(1) + \frac{1}{2} = 1$. Now, for the monotonicity, it is clear since f_n is monotonically increasing and from the definition of f_{n+1} combined with (5), that f_{n+1} is monotonically increasing. Finally, we prove that f_{n+1} is constant on $[0,1] \setminus P_{n+1}$. It is obvious from the definition that f_{n+1} is constant on $[1/3, 2/3]$. In addition, for any $x \in [0,1] \setminus P_{n+1}$, $3x \in [0,1] \setminus P_n$ if $0 \le x < 1/3$ and $3x - 2 \in [0,1] \setminus P_n$ if $2/3 \le x \le 1$. Since f_n is constant on $[0,1] \setminus P_n$, this implies that f_{n+1} is constant on $[0,1] \setminus P_{n+1}$.

(ii) As in the previous point, we use an induction argument. It is clear that

$$|f_1(x) - f_0(x)| < 1 \quad \forall x \in [0,1].$$

Now assume for some n that

$$|f_{n+1}(x) - f_n(x)| < 2^{-n} \quad \forall x \in [0,1].$$

and prove the inequality for $n+1$. For any $x \in [0, 1/3]$,

$$|f_{n+2}(x) - f_{n+1}(x)| = \frac{1}{2} |f_{n+1}(3x) - f_n(3x)| < 2^{-(n+1)}.$$

For any $x \in [2/3, 1]$,

$$|f_{n+2}(x) - f_{n+1}(x)| = \frac{1}{2} |f_{n+1}(3x-2) - f_n(3x-2)| < 2^{-(n+1)}$$

and for any $x \in [1/3, 2/3]$

$$|f_{n+2}(x) - f_{n+1}(x)| = 0 < 2^{-(n+1)}.$$

Thus,

$$|f_{n+2}(x) - f_{n+1}(x)| < 2^{-(n+1)} \quad \forall x \in [0,1].$$

which gives the desired inequality. This proves that $\{f_n\}_{n \in \mathbb{N}_0}$ is a uniformly converging Cauchy sequence. In particular, the pointwise limit of $f_n(x)$ exists for every $x \in [0,1]$ (since $\{f_n(x)\}$ is Cauchy on $\mathbb{R}$) and we call it $f(x)$. Since f_n are continuous on $[0,1]$

and by the above, they converge to f uniformly, we deduce that f is continuous on $[0,1]$.

(iii) The fact that f is monotonically increasing follows from the fact that every f_n is monotonically increasing. Indeed, assume the contrary, then there are $x < y$ such that $f(x) > f(y)$. Let $\varepsilon = f(x) - f(y)$. Since f_n converges uniformly to f there is an N such that for all $n \geq N$ $\|f_n - f\|_{C^0} < \varepsilon/5$. In particular, for N we get that

$$\begin{aligned} f_N(y) - f(y) &= f_N(y) - f_N(x) + f_N(x) - f(x) + f(x) - f(y) \\ &\geq f_N(x) - f(x) + f(x) - f(y) \geq \frac{4\varepsilon}{5}, \end{aligned}$$

which is a contradiction. Finally, we prove that f is piecewise constant on $[0,1] \setminus P$. Since f_n is piecewise constant on $[0,1] \setminus P_n$ and

$$P_{k+1} \subset P_k \quad \text{for every } k \geq 1,$$

for any n and all $m \geq n$, f_m is piecewise constant on $[0,1] \setminus P_n$. Since $f_n \to f$ uniformly, f is piecewise constant on $[0,1] \setminus P_n$. Since this is true for any n and

$$[0,1] \setminus P = \bigcup_{k=1}^{\infty} ([0,1] \setminus P_n),$$

f is piecewiece constant on $[0,1] \setminus P$.

(iv) It is enough to prove that $f(P) = [0,1]$. Since f is piecewise constant on $[0,1] \setminus P_n$, $f([0,1] \setminus P_n)$ is finite. Thus

$$\begin{aligned} f([0,1] \setminus P) = \{f(x) : x \in [0,1] \setminus P_n \text{ for some } n \geq 1\} &= \bigcup_{n=1}^{\infty} \{f(x) : x \in ([0,1] \setminus P_n)\} \\ = \bigcup_{n=1}^{\infty} f([0,1] \setminus P_n) \end{aligned}$$

is countable, and therefore $f(P)$ is dense in $[0,1]$. Since P is compact and f is continuous, also $f(P)$ is compact and hence $f(P) = \overline{f(P)} = [0,1]$.

Exercise 4. Show that there exists $f \colon [0,1] \to [0,1]$ continuous and two subsets $A, B \subseteq [0,1]$ such that

(i) A is measurable and $f(A)$ is not,

(ii) B is a null set and $f(B)$ has positive Lebesgue measure.

Hint: You can assume (without proof) that there exists a non-measurable subset $V \subset [0,1]$. An example of such a subset will be constructed explicitly in the lecture.

Solution:

(i) Let f be the Cantor function defined in Exercise 3 and $V \subset [0,1]$ be a non-measurable set (the Vitali set for example). Denote the Cantor set by P. Define

$$A = \{x \in P : f(x) \in V\}.$$

Since $A \subseteq P$ and the Cantor set P has Lebesgue measure 0, A has outer measure 0 and is therefore measurable (of measure 0). Since f maps P to $[0,1]$ surjectively, $f(A) = V$ and therefore $f(A)$ is not measurable.

(ii) Again, let f be the Cantor function and set $B = P$ to be the Cantor set P. We already know that $B = P$ has measure 0, and $f(B) = [0,1]$, which has measure 1.

Exercise 5. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$. Prove the following statements:

(i) There exists a positive continuous function $f : \mathbb{R} \to [0, +\infty)$ which is absolutely integrable and yet $\limsup_{x\to\infty} f(x) = \infty$.

(ii) If f is absolutely integrable and $\lim_{|x|\to\infty} f(x)$ exists, then necessarily $\lim_{|x|\to\infty} f(x) = 0$.

(iii) Show that if f is uniformly continuous and absolutely integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Solution:

(i) For each $n \in \mathbb{N}^*$, we define $\varphi_n \colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi_n(x) = \begin{cases} 3n^3(x-n) & \text{if } n \le x \le n + \frac{1}{3n^3}, \\ 1 & \text{if } n + \frac{1}{3n^3} \le x \le n + \frac{2}{3n^3}, \\ 3n^3\left(n + \dfrac{1}{n^3} - x\right) & \text{if } n + \frac{2}{3n^3} \le x \le n + \frac{1}{n^3}, \\ 0 & \text{otherwise.} \end{cases}$$

By construction $\int_{\mathbb{R}} |\varphi_n|\, dx = \int_{\mathbb{R}} \varphi_n\, dx \le n^{-3}$ and φ_n is continuous. Now, define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} n\varphi_n(x)$$

and observe that f is integrable and continuous. However, we have

$$\limsup_{x\to\infty} f(x) = \lim_{n\to\infty} n = \infty.$$

(ii) Assume for a contradiction that $\lim_{|x|\to\infty} f(x) = \delta \neq 0$. Then there is $R > 0$ such that for all $x \in \mathbb{R}$ with $|x| > R$ we have $|f(x)| > 2^{-1}|\delta|$. Thus,

$$\int_{\mathbb{R}} |f(x)|\,dx \geq \int_{|x|>R} \frac{|\delta|}{2}\,dx = \infty$$

so that f is not integrable, which yields the desired contradiction.

(iii) Assume for a contradiction that $f(x)$ does not converge to 0 as $|x| \to \infty$. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $|x_n| \to \infty$ and $\lim_{n\to\infty} |f(x_n)| = c > 0$. We can assume without loss of generality that $|x_{n+1}| \geq |x_n| + 1$ for all $n \geq 1$. There is $N \geq 1$ such that for all $n \geq N$ $|f(x_n)| > c/2$. By uniform continuity, there is $0 < \delta < 1$ such that whenever x and y satisfy $|x - y| < \delta$, we get $|f(x) - f(y)| < c/4$. Thus, for any $n \geq N$ and any $x \in (x_n - \delta, x_n + \delta)$, we have

$$|f(x)| > c/4.$$

Finally, since the intervals $\{(x_n - \delta, x_n + \delta)\}_{n\geq N}$ are disjoint, we get

$$\int_{\mathbb{R}} |f(x)|\,dx \geq \int_{\cup_{n\geq N}(x_n-\delta,x_n+\delta)} |f(x)|\,dx \geq \sum_{n\geq N} \int_{x_n-\delta}^{x_n+\delta} |f(x)|\,dx \geq \sum_{n\geq N} \frac{c\delta}{2} = \infty,$$

which shows that f is not integrable and gives us a contradiction.

Exercise 6 ($\star$)**.** Let $0 < \varepsilon < 1$.

(i) Construct an open dense set $E \subseteq [0,1]$ such that $\mathrm{m}(E) = \varepsilon$.

(ii) Construct a closed set F that does not contain any non-empty open set with $\mathrm{m}(F) = \varepsilon$.

Solution:

(i) We construct the set E as an countable union of open and pairwise disjoint sets $\{E_k\}_{k=0}^{\infty}$. We will inductively construct the sets $\{E_k\}_{k=1}^{\infty}$ having the properties

(a) $\mathrm{m}(E_k) = \dfrac{\varepsilon}{2^{k+1}}$,

(b) $E_i \cap E_j = \emptyset \ \forall i \neq j$,

(c) E_k is the union of 2^k disjoint, open intervals of equal length $\frac{\varepsilon}{2^{2(k+1)}}$,

(d) $\left(\bigcup_{i=0}^{k-1} E_i\right)^c$ is a union of 2^k disjoint closed sets of equal length $\big(1 - \varepsilon\big(1 - \dfrac{1}{2^k}\big)\big)$.

We start the construction by defining

$$E_0 := \left(\frac{1}{2} - \frac{\varepsilon}{4}, \frac{1}{2} + \frac{\varepsilon}{4}\right),$$

Now, assume by induction that we have already constructed the sets $E_0, \ldots, E_{n-1}$ verifying (a)-(d) for some $n \geq 1$. Now we will construct the set E_n. Let $x_1, \ldots, x_{2^n}$ be the middle points of the complementaries of those intervals. Since

$$\frac{\varepsilon}{2^{2n+1}} \leq \frac{1}{2^n}\left(1 - \varepsilon\left(1 - \frac{1}{2^n}\right)\right),$$

the sets

$$\left(x_i - \frac{\varepsilon}{2^{2n+2}}, x_i + \frac{\varepsilon}{2^{2n+2}}\right)$$

are clearly disjoint from $\bigcup_{k=1}^{n-1} E_k$. Defining,

$$E_n = \bigcup_{i=1}^{2^n} \left(x_i - \frac{\varepsilon}{2^{2n+2}}, x_i + \frac{\varepsilon}{2^{2n+2}}\right),$$

it is clear that $E_i \cap E_j = \emptyset$, $\forall i, j = 1, \ldots, n$, $i \neq j$. In addition,

$$\mathrm{m}(E_n) = 2^n \frac{2\varepsilon}{2^{2n+2}} = \frac{\varepsilon}{2^{n+1}}.$$

Finally, the complement of $\cup_{k=1}^{n} E_k$ consists of 2^{n+1} intervals since every interval in $(\cup_{k=1}^{n-1} E_k)^c$ generates two intervals in $(\cup_{k=1}^{n} E_k)^c$ due to the construction of E_n. From this construction, it is clear that all these intervals have length

$$\frac{\frac{1}{2^n}(1 - \varepsilon(1 - \frac{1}{2^n})) - \frac{\varepsilon}{2^{2n+1}}}{2} = \frac{1}{2^{n+1}}\left(1 - \varepsilon\left(1 - \frac{1}{2^{n+1}}\right)\right).$$

This proves that the family $\{E_k\}_{k=1}^{\infty}$ is composed of disjoint sets and

$$\mathrm{m}(E) = \mathrm{m}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathrm{m}(E_k) = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon.$$

Now we prove that E is dense in $[0, 1]$. Let $x \in [0, 1]$ and $\delta > 0$ and prove that there is $y \in E$ such that $|x - y| < \delta$. If $x \in E$, it is obvious. Instead, we suppose that $x \notin E$. Then there is $n \geq 1$ such that

$$\frac{1}{2^{n+1}} < \delta.$$

Since $x \notin E$, $x \notin \bigcup_{i=1}^{n} E_i$, i.e. $x \in (\bigcup_{i=1}^{n} E_i)^c$. Since $(\bigcup_{i=1}^{n} E_i)^c$ consists of disjoint intervals of length

$$\frac{1}{2^{n+1}}\left(1 - \varepsilon\left(1 - \frac{1}{2^{n+1}}\right)\right) < \frac{1}{2^{n+1}},$$

there must exist $y \in \bigcup_{i=1}^{n} E_i$ such that

$$|x - y| < \frac{1}{2^{n+1}} < \delta.$$

This proves that E is dense in $[0,1]$.

(ii) Due to the previous point, there is an open, dense set $E \subseteq [0,1]$ such that $\mathrm{m}(E) = 1-\varepsilon$. Define $F = [0,1] \setminus E$. Thus

$$\mathrm{m}(F) = 1 - \mathrm{m}(E) = \varepsilon.$$

Since E is dense in $[0,1]$, F cannot contain a non-empty open set.