Serie 3

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Definition 1 (lower/upper semi-continuity)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function.

(i) f is *lower semi-continuous* in $x_0 \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \ \text{ such that } \ |x - x_0| < \delta \ \Rightarrow \ f(x_0) - f(x) \le \varepsilon.$$

(ii) f is *lower semi-continuous* if f is lower semi-continuous in every point $x_0 \in \mathbb{R}^n$.

(iii) f is *upper semi-continuous* in $x_0 \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \ \text{ such that } \ |x - x_0| < \delta \ \Rightarrow \ f(x) - f(x_0) \le \varepsilon.$$

(iv) f is *upper semi-continuous* if f is upper semi-continuous in every point $x_0 \in \mathbb{R}^n$.

Exercise 1. We show that lower/upper semi-continuity implies measurability.

(i) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous, then for all $\alpha \in \mathbb{R}$ the set

$$G_\alpha := \{x \in \mathbb{R} : f(x) \le \alpha\}$$

is closed. Similarly, show that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is upper semi-continuous, then for all $\alpha \in \mathbb{R}$ the set

$$F_\alpha := \{x \in \mathbb{R} : f(x) \ge \alpha\}$$

is closed.

(ii) Deduce that an lower/upper semi-continuous function is measurable.

Exercise 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be increasing or decreasing. Prove that f is measurable.

Exercise 3. Let $f,\, g\colon \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Show that the functions

$$f^2, \quad fg, \quad |f|$$

are measurable.

Exercise 4. Let φ be measurable and f continuous. Show that $f \circ \varphi$ is measurable. (On the other hand, in general $\varphi \circ f$ is not measurable and we will discuss a counterexample in Serie 5.)

Exercise 5. Let $\Omega \subseteq \mathbb{R}^n$ measurable and let $f : \Omega \to [0, \infty)$ be a nonnegative and integrable function. If $\alpha > 0$ and $E_\alpha := \{x \in \Omega : f(x) > \alpha\}$, prove that

$$\mathrm{m}(E_\alpha) \leq \frac{1}{\alpha} \int_\Omega f\, dx.$$

Exercise 6. Let $\Omega \subset \mathbb{R}^n$ measurable and let $f\colon \Omega \to \mathbb{R}$ be integrable. Show that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any measurable set $E \subset \Omega$, it holds that if

$$\mathrm{m}(E) \leq \delta \;\Rightarrow\; \int_E |f(x)|\, dx \leq \varepsilon.$$

Hint: Consider the sequence $f_n(x) := \min\{|f(x)|, n\}$.

Exercise 7 ($\star$)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ continuous. Prove that the sets of points $x \in \mathbb{R}$ where f is differentiable is a Lebesgue measurable set.