Serie 3

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Definition 1 (lower/upper semi-continuity)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function.

(i) f is *lower semi-continuous* in $x_0 \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ such that } |x - x_0| < \delta \;\Rightarrow\; f(x_0) - f(x) \leq \varepsilon.$$

(ii) f is *lower semi-continuous* if f is lower semi-continuous in every point $x_0 \in \mathbb{R}^n$.

(iii) f is *upper semi-continuous* in $x_0 \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ such that } |x - x_0| < \delta \;\Rightarrow\; f(x) - f(x_0) \leq \varepsilon.$$

(iv) f is *upper semi-continuous* if f is upper semi-continuous in every point $x_0 \in \mathbb{R}^n$.

Exercise 1. We show that lower/upper semi-continuity implies measurability.

(i) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous, then for all $\alpha \in \mathbb{R}$ the set

$$G_\alpha := \{x \in \mathbb{R} : f(x) \leq \alpha\}$$

is closed. Similarly, show that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is upper semi-continuous, then for all $\alpha \in \mathbb{R}$ the set

$$F_\alpha := \{x \in \mathbb{R} : f(x) \geq \alpha\}$$

is closed.

(ii) Deduce that an lower/upper semi-continuous function is measurable.

Solution:

(i) We show that if f is lower semi-continuous, then G_α is closed. Let $\{x_n\}_{n=1}^{\infty} \subset G_\alpha$ such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}^n$. We need to show that $x \in G_\alpha$. Let $\varepsilon > 0$. By lower semi-continuity in x, there is $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow f(x) \leq f(y) + \varepsilon.$$

Since $x_n \to x$ as $n \to \infty$, there is $N = N(\delta) \in \mathbb{N}$ such that $|x - x_n| \leq \delta$ for all $n \geq N$. Hence,

$$f(x) \leq f(x_n) + \varepsilon \leq \alpha + \varepsilon, \quad \forall n \geq N.$$

The previous inequality being true for any ε, we conclude

$$f(x) \leq \alpha.$$

Thus, $x \in G_\alpha$ and therefore G_α is closed. In order to prove that if f is upper semi-continuous, then F_α is closed, we use the same approach.

(ii) If f is lower semi-continuous, then G_α is closed and therefore measurable for every $\alpha \in \mathbb{R}$. We deduce that f is measurable. Finally, if f is upper semi-continuous, the measurability follows from the fact that F_α closed for every $\alpha \in \mathbb{R}$.

Exercise 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be increasing or decreasing. Prove that f is measurable.

Solution: Assume that f is increasing. We show that for any $\alpha \in \mathbb{R}$

$$E_\alpha := \{x \in \mathbb{R} : f(x) > \alpha\}$$

is measurable. If $E_\alpha = \emptyset$ the measurability is trivial. If $E_\alpha \neq \emptyset$, we set $a := \inf E_\alpha \in \mathbb{R} \cup \{-\infty\}$. Note that if $x \in E_\alpha$, then $x \geq a$ and for any $y \geq x$, we have $y \in E_\alpha$ (since f is increasing). Thus, $[x, \infty[\subset E_\alpha$. Therefore E_α is either equal to $\emptyset$, $]a, \infty[$, $[a, \infty[$ (for $a \in \mathbb{R}$) or to $\mathbb{R}$. In any case E_α is measurable and hence f is measurable. The argument when f is decreasing is similar.

Exercise 3. Let f, $g\colon \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Show that the functions

$$f^2, \quad fg, \quad |f|$$

are measurable.

Solution:

(i) We begin with f^2: We show that for any $\alpha \in \mathbb{R}$, the set $E_\alpha := \{x \in \mathbb{R}^n : f^2(x) > \alpha\}$ is measurable. If $\alpha < 0$, $E_\alpha = \mathbb{R}$. On the other hand, if $\alpha \geq 0$

$$E_\alpha = \{x \in \mathbb{R}^n : f(x) > \sqrt{\alpha}\} \cup \{x \in \mathbb{R}^n : f(x) < -\sqrt{\alpha}\}$$

is measurable, since the union of two measurable sets is measurable.

(ii) Now fg: Note that

$$fg = \frac{1}{2}\{(f+g)^2 - f^2 - g^2\}.$$

Using (i), $(f+g)^2$, f^2, and g^2 are measurable. Thus, fg is measurable since it can be written as a finite sum of measurable functions.

(iii) Finally $|f|$: We show that $E_\alpha := \{x \in \mathbb{R}^n : |f(x)| > \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$. When $\alpha < 0$, $E_\alpha = \mathbb{R}$. When $\alpha \geq 0$, we have that

$$E_\alpha = \{x \in \mathbb{R}^n : |f(x)| > \alpha\} = \{x \in \mathbb{R}^n : f(x) > \alpha\} \cup \{x \in \mathbb{R}^n : f(x) < -\alpha\}$$

is measurable as finite union of measurable sets.

Exercise 4. Let φ be measurable and f continuous. Show that $f \circ \varphi$ is measurable. (On the other hand, in general $\varphi \circ f$ is not measurable and we will discuss a counterexample in Serie 5.)

Solution: We is enough to show that for any $\alpha \in \mathbb{R}$ the set

$$E_\alpha := \{x \in \mathbb{R} : (f \circ \varphi)(x) > \alpha\},$$

is measurable. We have

$$E_\alpha = (f \circ \varphi)^{-1}(]\alpha, \infty[) = \varphi^{-1}(f^{-1}(]\alpha, \infty[)).$$

Using the continuity of f, $O := f^{-1}(]\alpha, \infty[)$ is open and therefore $\varphi^{-1}(O)$ is measurable by the measurability of φ.

Exercise 5. Let $\Omega \subseteq \mathbb{R}^n$ measurable and let $f : \Omega \to [0, \infty)$ be a nonnegative and integrable function. If $\alpha > 0$ and $E_\alpha := \{x \in \Omega : f(x) > \alpha\}$, prove that

$$\mathrm{m}(E_\alpha) \leq \frac{1}{\alpha} \int_\Omega f \, dx.$$

Solution: Define the function $g_\alpha : \Omega \to \mathbb{R}$ by

$$g_\alpha(x) = \alpha \chi_{E_\alpha}(x) = \begin{cases} \alpha & \text{if } x \in E_\alpha, \\ 0 & \text{if } x \notin E_\alpha. \end{cases}$$

Observe that g_α is measurable because the set E_α is a Lebesgue-measurable set (which, in turn, is a consequence of the measurability of f). Moreover $g_\alpha \leq f$ pointwise and therefore by the monotonicity of the integral

$$\alpha \cdot \mathrm{m}(E_\alpha) = \int_\Omega g_\alpha \, dx \leq \int_\Omega f \, dx.$$

Exercise 6. Let $\Omega \subset \mathbb{R}^n$ measurable and let $f \colon \Omega \to \mathbb{R}$ be integrable. Show that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any measurable set $E \subset \Omega$, it holds that if

$$\mathrm{m}(E) \leq \delta \ \Rightarrow \ \int_E |f(x)| \, dx \leq \varepsilon.$$

Hint: Consider the sequence $f_n(x) := \min\{|f(x)|, n\}$.

Solution: For $\nu \in \mathbb{N}$ we set

$$f_\nu(x) := \min\{|f(x)|, \nu\}.$$

The sequence if f_ν is an increasing sequence of measurable, nonnegative functions. Moreover, $f_\nu \to |f|$ pointwise as $\nu \to \infty$. The monotone convergence theorem implies that

$$\lim_{\nu \to \infty} \int_\Omega f_\nu \, dx = \int_\Omega |f(x)| \, dx.$$

In particular, for every $\varepsilon > 0$, there is $\nu = \nu(\delta)$ large enough such that

$$\int_\Omega (|f(x)| - f_\nu(x)) \, dx \leq \frac{\varepsilon}{2}.$$

We define $\delta = \varepsilon/2\nu$. Then, for any measurable set $E \subset \Omega$ such that $\mathrm{m}(E) \leq \delta$, we have

$$\int_E |f(x)| \, dx = \int_E f_\nu(x) \, dx + \int_E (|f(x)| - f_\nu(x)) \, dx \leq \nu \, \mathrm{m}(E) + \frac{\varepsilon}{2} \leq \varepsilon.$$

Exercise 7 ($\star$). Let $f \colon \mathbb{R} \to \mathbb{R}$ continuous. Prove that the sets of points $x \in \mathbb{R}$ where f is differentiable is a Lebesgue measurable set.

Solution: The solution is divided in several steps.

Step 1: We begin by proving the following equivalence

$$f \text{ is differentiable at } x_0 \quad \Leftrightarrow \quad \forall n \in \mathbb{N}, \exists m \in \mathbb{N} \text{ such that } \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n}$$
$$\text{for all } x, y \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m}\right) \text{ and } x \neq x_0 \neq y.$$

We begin with the direction $\Rightarrow$ which is the easiest one. Let $n \in \mathbb{N}$ be arbitrary. By differentiability, there exists m such that for all $x \in (x_0 - \frac{1}{m}, x_0 + \frac{1}{m})$, $x \neq x_0$

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \frac{1}{2n}.$$

Thus for all $x, y \in (x_0 - \frac{1}{m}, x_0 + \frac{1}{m})$, $x \neq x_0$, $y \neq x_0$ we have by the triangular inequality that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n}.$$

Now we prove the reverse direction $\Leftarrow$. We will show that the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists.} \tag{1}$$

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence such that $x_k \to x_0$. It is clear that $\dfrac{f(x_k) - f(x_0)}{x_k - x_0}$ is a Cauchy sequence and therefore it converges. In order to prove that the above limit in (**??**) exists, we need to show that the limit of the sequence $\dfrac{f(x_k) - f(x_0)}{x_k - x_0}$ is independent of the choice of the sequence $\{x_k\}_{k=1}^{\infty}$. Again, this follows easily from the condition that we suppose to be true.

Step 2: We rewrite the set of points of differentiability of f.

From Step 1,

$$\begin{aligned}
&\{x_0 \in \mathbb{R} : f \text{ is differentiable in } x_0\} \\
&= \left\{ x_0 \in \mathbb{R} : \forall n \in \mathbb{N}, \exists m \in \mathbb{N} \text{ such that } \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n} \right. \\
&\qquad \left. \text{for all } x, y \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m} \right) \text{ and } x \neq x_0 \neq y. \right\} \\
&= \bigcap_{n \in \mathbb{N}} \left\{ x_0 \in \mathbb{R} : \exists m \in \mathbb{N} \text{ such that } \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n} \right. \\
&\qquad \left. \text{for all } x, y \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m} \right) \text{ and } x \neq x_0 \neq y. \right\} \\
&= \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \left\{ x_0 \in \mathbb{R} : \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n} \right. \\
&\qquad \left. \text{for all } x, y \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m} \right) \text{ and } x \neq x_0 \neq y. \right\}.
\end{aligned}$$

We define the sets $E_{n,m}$ as

$$E_{n,m} = \left\{ x_0 \in \mathbb{R} : \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n}, \forall x, y \in \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m} \right), x \neq x_0 \neq y \right\}.$$

With this notation

$$\{x_0 \in \mathbb{R} : f \text{ is differentiable in } x_0\} = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} E_{n,m}.$$

Step 3: In order to prove that $\{x_0 \in \mathbb{R} : f$ is differentiable in $x_0\}$ is measurable, we will prove that each $E_{n,m}$ is closed.

Let $\{x_0^i\} \subseteq E_{n,m}$ be a sequence such that $x_0^i \to x_0 \in \mathbb{R}$ as $i \to \infty$. Let $x, y \in (x_0 - \frac{1}{m}, x_0 + \frac{1}{m})$, $x, y \neq x_0$ then there is N such that $\forall i \geq N$, $x, y \in (x_0^i - \frac{1}{m}, x_0^i + \frac{1}{m})$, $x, y \neq x_0^i$. Thus,

$$\left| \frac{f(x) - f(x_0^i)}{x - x_0^i} - \frac{f(y) - f(x_0^i)}{y - x_0^i} \right| \leq \frac{1}{n}.$$

Letting $i \to \infty$ und using the continuity of f, we deduce

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(y) - f(x_0)}{y - x_0} \right| \leq \frac{1}{n},$$

hence $x_0 \in E_{n,m}$. We conclude that $E_{n,m}$ is closed.