Serie 2
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday at 8am, midnight, via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $A, B \subset \mathbb{R}^d$. Show that if $A \subseteq B$ and $\mathrm{m}^*(B) = 0$, then $\mathrm{m}^*(A) = 0$.

Solution: Let $\varepsilon > 0$. By definition of the outer Lebesgue measure, there is a countable collection of cubes $\{B_j\}_{j\in J}$ covering B such that

$$\sum_{j\in J} \mathrm{vol}(B_j) \leq \mathrm{m}^*(B) + \varepsilon = \varepsilon.$$

Since $A \subseteq B$, $\{B_j\}_{j\in J}$ covers A, and hence, from the definiton of outer measure

$$0 \leq \mathrm{m}^*(A) \leq \varepsilon.$$

Since the previous inequality is true for any ε, we conclude $\mathrm{m}^*(A) = 0$.

Exercise 2. If $A \subseteq \mathbb{R}^d$ and E is the half-plane $E := \{(x_1, \dots, x_n) \in \mathbb{R}^d : x_n > 0\}$, show that $\mathrm{m}^*(A) = \mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E)$.

Solution: We begin by proving $\mathrm{m}^*(A) \leq \mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E)$. For any $\varepsilon > 0$, let $\{B_i^1\}_{i=1}^\infty$, $\{B_i^2\}_{i=1}^\infty$ be countable families of open boxes such that

$$\sum_{i=1}^\infty \mathrm{vol}(B_i^1) \leq \mathrm{m}^*(A \cap E) + \varepsilon, \quad A \cap E \subseteq \bigcup_{i=1}^\infty B_i^1$$

and

$$\sum_{i=1}^\infty \mathrm{vol}(B_i^2) \leq \mathrm{m}^*(A \setminus E) + \varepsilon, \quad A \setminus E \subseteq \bigcup_{i=1}^\infty B_i^2.$$

Then the countable collection $\{B_i\}_{i=1}^{\infty}$ defined by

$$B_i \colon = \begin{cases} B^1_{i/2} & \text{if } i \text{ is even,} \\ B^2_{(i+1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

covers A, hence

$$\mathrm{m}^*(A) \leq \sum_{i=1}^{\infty} |B_i| \leq \sum_{i=1}^{\infty} |B_i^1| + \sum_{i=1}^{\infty} |B_i^2| \leq \mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E) + 2\varepsilon.$$

The previous inequality being true for any $\varepsilon > 0$, we deduce

$$\mathrm{m}^*(A) \leq \mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E).$$

Then we prove the reverse inequality. Fix $\varepsilon > 0$. There is a countable family of open boxes $\{B_i\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} \mathrm{vol}(B_i) \leq \mathrm{m}^*(A) + \varepsilon, \quad A \subseteq \bigcup_{i=1}^{\infty} B_i.$$

Every B_i is an open box of the form $B_i = \prod_{k=1}^n (a_k^{(i)}, b_k^{(i)})$. Define for any $i = 1, 2, \ldots$,

$$\varepsilon_i = \frac{\varepsilon}{2^i \prod_{k=1}^{n-1} |a_k^{(i)} - b_k^{(i)}|},$$

and correspondingly,

$$B_i^1 = \prod_{k=1}^{n-1} (a_k^{(i)}, b_k^{(i)}) \times \left[(a_n^{(i)}, b_n^{(i)}) \cap (0, \infty) \right],$$
$$B_i^2 = \prod_{k=1}^{n-1} (a_k^{(i)}, b_k^{(i)}) \times \left[(a_n^{(i)}, b_n^{(i)}) \cap (-\infty, \varepsilon_i) \right].$$

Notice that B_i^1 and B_i^2 are open boxes and

$$\mathrm{vol}(B_i^1) + \mathrm{vol}(B_i^2) \leq \prod_{k=1}^{n-1} |a_k^{(i)} - b_k^{(i)}| \left(|a_n^{(i)} - b_n^{(i)}| + \varepsilon_i \right) = \prod_{k=1}^{n} |a_k^{(i)} - b_k^{(i)}| + \varepsilon/2^i = \mathrm{vol}(B_i) + \varepsilon/2^i$$

We can prove that $A \cap E \subseteq \cup_{i=1}^{\infty} B_i^1$ and $A \setminus E \subseteq \cup_{i=1}^{\infty} B_i^2$. Finally,

$$\mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E) \leq \sum_{i=1}^{\infty} \mathrm{vol}(B_i^1) + \sum_{i=1}^{\infty} \mathrm{vol}(B_i^2) \leq \sum_{i=1}^{\infty} \left[\mathrm{vol}(B_i) + \varepsilon/2^i \right] \leq \mathrm{m}^*(A) + 2\varepsilon.$$

Since this inequality is true for any ε, we deduce

$$\mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E) \leq \mathrm{m}^*(A).$$

Exercise 3. We want to show that the notion of Lebesgue outer measure does not depend on whether we consider coverings by open, half-open or closed boxes. To this end, we introduce for a subset $\Omega \subseteq \mathbb{R}^n$

$$\overline{\mathrm{m}}^*(\Omega) = \inf \left\{ \sum_{j=1}^{\infty} \mathrm{vol}(B_j) : \quad \Omega \subseteq \bigcup_{j=1}^{\infty} B_j \text{ and for all } j \geq 1 \quad B_j = \prod_{i=1}^{n} [a_i^{(j)}, b_i^{(j)}] \right.$$
$$\left. \text{for some } -\infty < a_i^{(j)} < b_i^{(j)} < +\infty \right\}.$$

Show that for any subset $\Omega \subseteq \mathbb{R}^n$

$$\overline{\mathrm{m}}^*(\Omega) = \mathrm{m}^*(\Omega).$$

Solution: Let $\Omega \subseteq \mathbb{R}^n$. We first prove $\overline{\mathrm{m}}^*(\Omega) \leq \mathrm{m}^*(\Omega)$. Fix $\varepsilon > 0$. There exists a cover of Ω consisting of countably many open boxes $B_j = \prod_{i=1}^n (a_i^{(j)}, b_i^{(j)})$ such that

$$\sum_{j=1}^{\infty} \mathrm{vol}(B_j) \leq \mathrm{m}^*(\Omega) + \varepsilon\,.$$

Then consider the collection of closed boxes $\{\widetilde{B_j}\}_{j=1}^{\infty}$ given by the closure of the B_j, that is $\widetilde{B_j} := \prod_{i=1}^n [a_i^{(j)}, b_i^{(j)}]$. Note that $\mathrm{vol}(\widetilde{B_j}) = \mathrm{vol}(B_j)$, $B_j \subseteq B_j$ and hence $\Omega \subseteq \bigcup_{j=1}^{\infty} \widetilde{B_j}$, so that

$$\overline{\mathrm{m}}^*(\Omega) \leq \sum_{j=1}^{\infty} \mathrm{vol}(\widetilde{B_j}) = \sum_{j=1}^{\infty} \mathrm{vol}(B_j) \leq \mathrm{m}^*(\Omega) + \varepsilon.$$

Since ε is arbitrary, we conclude

$$\overline{\mathrm{m}}^*(\Omega) \leq \mathrm{m}^*(\Omega). \tag{1}$$

We now show the reverse inequality. Fix $\varepsilon > 0$. There exists a countable cover of Ω consisting of closed boxes $\widetilde{B_j} = \prod_{i=1}^n [a_i^{(j)}, b_i^{(j)}]$ such that

$$\sum_{j=1}^{\infty} \mathrm{vol}(\widetilde{B_j}) \leq \overline{\mathrm{m}}^*(\Omega) + \varepsilon\,.$$

Now for each $j \geq 1$, define $\varepsilon^{(j)} = (\sqrt[n]{1+\varepsilon} - 1) \min_{i=1,\ldots,n}\{b_i^{(j)} - a_i^{(j)}\}$. Consider the collection of open boxes $\{B_j\}_{j=1}^{\infty}$ defined by

$$B_j = \prod_{i=1}^{n} \big(a_i^{(j)} - \frac{\varepsilon^{(j)}}{2}, b_i^{(j)} + \frac{\varepsilon^{(j)}}{2}\big).$$

Then the B_j are open boxes which cover Ω (since $\tilde{B}_j \subset B_j$ for all $j \geq 1$) and

$$\mathrm{vol}(B_j) = \prod_{i=1}^{n} (b_i^{(j)} - a_i^{(j)} + 2\varepsilon^{(j)}) \leq \prod_{i=1}^{n} \sqrt[n]{1+\varepsilon}(b_i^{(j)} - a_i^{(j)}) = (1+\varepsilon)\,\mathrm{vol}(\widetilde{B_j})\,.$$

Thus,

$$\mathrm{m}^*(\Omega) \leq \sum_{j=1}^{\infty} \mathrm{vol}(B_j) \leq (1+\varepsilon) \sum_{j=1}^{\infty} \mathrm{vol}(\widetilde{B_j}) \leq (1+\varepsilon)\overline{\mathrm{m}}^*(\Omega) + \varepsilon.$$

Since ε is arbitrary, we deduce

$$\mathrm{m}^*(\Omega) \leq \overline{\mathrm{m}}^*(\Omega). \tag{2}$$

Finally, we deduce the equality from (1) and (2).

Exercise 4. We want to compute the measure of the intersection of a countable family of decreasing sets.

(i) Show that if $A_1 \supseteq A_2 \supseteq A_3 \dots$ is a decreasing sequence of measurable sets (that is $A_j \supseteq A_{j+1}$ for every $j \geq 1$) and $\mathrm{m}(A_1) < +\infty$, then

$$\mathrm{m}\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mathrm{m}(A_j)$$

(ii) Show that the previous equality fails without the assumption $\mathrm{m}(A_1) < +\infty$.

Solution:

(i) Note that the sets $A_j \setminus A_{j+1}$, $j = 1, 2, \dots$, are disjoint and in addition they are disjoint from the set $\bigcap_{j=1}^{\infty} A_j$. Moreover, using since A_j are decreasing, we can write

$$\bigcap_{j=1}^{\infty} A_j \cup \bigcup_{j=1}^{\infty} [A_j \setminus A_{j+1}] = A_1,$$

and therefore

$$\mathrm{m}\left(\bigcap_{j=1}^{\infty} A_j\right) + \mathrm{m}\left(\bigcup_{j=1}^{\infty} A_j \setminus A_{j+1}\right) = \mathrm{m}(A_1). \tag{3}$$

Since $\mathrm{m}(A_1) < \infty$, $\mathrm{m}(A_k) < \infty$ for any $k \geq 1$. As a consequence,

$$\mathrm{m}(A_j \setminus A_{j+1}) = \mathrm{m}(A_j) - \mathrm{m}(A_{j+1}) \text{ for any } j \geq 1\,.$$

Thus,

$$\mathrm{m}\left(\bigcup_{j=1}^{\infty} A_j \setminus A_{j+1}\right) = \sum_{j=1}^{\infty} \mathrm{m}(A_j \setminus A_{j+1}) = \lim_{k\to\infty} \sum_{j=1}^{k} \mathrm{m}(A_j \setminus A_{j+1}) = \lim_{k\to\infty} \sum_{j=1}^{k} [\mathrm{m}(A_j) - \mathrm{m}(A_{j+1})]$$
$$= \mathrm{m}(A_1) - \lim_{k\to\infty} \mathrm{m}(A_k).$$

From this last equality and (3), we deduce the result.

(ii) Observe that from (i), any counter-example must satisfy $\mathrm{m}(A_k) = \infty$ for all $k \geq 1$. If we take, for instance, $A_k = (k, \infty)$, then $\bigcap_{j=1}^{\infty} A_j = \emptyset$, so that

$$\mathrm{m}\left(\bigcap_{j=1}^{\infty} A_j\right) = 0.$$

However, since $\mathrm{m}(A_k) = \infty$ for all $k = 1, 2, 3, \ldots$, we have $\lim_{j\to\infty} \mathrm{m}(A_j) = \infty$.

The following technical exercise will be used, in a future exercise sheet, to give an equivalent definition of the Cantor set and to prove some of its interesting properties.

Exercise 5. For any sequence $\{a_i\}_{i=1}^{\infty}$ with $a_i \in \{0, 1, 2\}$, we denote by $0.a_1a_2a_3\ldots$ the number

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i}.$$

Consider the inductive construction described below: for all $a \in [0, 1]$, we define

$$\begin{cases} a_1 := \lfloor 3a \rfloor, \\ a_{i+1} := \lfloor 3^{i+1}(a - \widetilde{a}_i) \rfloor \quad i \geq 1, \end{cases}$$

where for $i \geq 1$

$$\widetilde{a}_i := \sum_{n=1}^{i} \frac{a_n}{3^n}.$$

Here we set

$$\lfloor y \rfloor := \begin{cases} \max\{n \in \mathbb{N} : n < y\} & \text{if } y > 0, \\ 0 & \text{if } y = 0. \end{cases}$$

(i) Show that for all $i \geq 1$

$$0 \leq a - \widetilde{a}_i \leq \frac{1}{3^i}.$$

Deduce that any $a \in [0, 1]$ can be written as $0.a_1a_2a_3\ldots$ with $a_i \in \{0, 1, 2\}$.

(ii) Conversly, assume that $\{a_i\}_{i=1}^{\infty}$ is a sequence with $a_i \in \{0, 1, 2\}$ for all $i \geq 1$. Show that $0.a_1a_2a_3\ldots \in [0, 1]$.

(iii) The expansion of a number $a \in [0, 1]$ as $0.a_1a_2a_3\ldots$ is called the ternary expansion. Show that, in general, this expansion is not unique.

(iv) We now adopt the following identification among tenary expansions: if there exists $k \geq 2$ such that $a_i = 2$ for all $i \geq k$ and $a_{i-1} < 2$, then we identify the expansion

$$0.a_1 \cdots a_{k-1} 2 \cdots 2 \ldots$$

with the expansion

$$0.a_1 \cdots (a_{k-1} + 1) 0 \cdots 0 \ldots$$

Prove by contradiction that, modulo this identification, the ternary expansion of a number $a \in [0, 1]$ is unique.

Hint: For (iii), recall that $1 = 0.9999\ldots$

Solution:

(i) The solution is divided into 3 steps. Let $a \in [0,1]$ and let $\{a_i\}_{i\in\mathbb{N}}$ be the sequence defined in the exercise.

Step 1: We show that for all $i \geq 1$ we have

$$0 \leq a - \widetilde{a}_i \leq \frac{1}{3^i}. \tag{4}$$

.

We know that for all $x \geq 0$,

$$x - \lfloor x \rfloor \in [0,1].$$

For $i = 1$, we have

$$a - \widetilde{a}_1 = a - \frac{\lfloor 3a \rfloor}{3} = \frac{1}{3}(3a - \lfloor 3a \rfloor) \in [0, 1/3].$$

For $i \geq 2$, we compute

$$3^i(a - \widetilde{a}_i) \quad = 3^i\left(a - \widetilde{a}_{i-1} - \frac{a_i}{3^i}\right) = 3^i(a - \widetilde{a}_{i-1}) - \lfloor 3^i(a - \widetilde{a}_{i-1}) \rfloor \quad \in [0,1]\,.$$

Step 2: We show that $a_i \in \{0,1,2\}$.

Indeed, since $0 \leq a \leq 1$, we have $0 \leq 3a \leq 3$ and thus $a_1 = \lfloor 3a \rfloor \in \{0,1,2\}$. For $i > 1$ we use (4) to obtain $0 \leq 3^i(a - \widetilde{a}_i) \leq 1$ and thus

$$0 \leq 3^{i+1}(a - \widetilde{a}_i) \leq 3, \text{ which implies } a_{i+1} = \lfloor 3^{i+1}(a - \widetilde{a}_i) \rfloor \in \{0,1,2\}.$$

Step 3: Finally, we show that $a = 0.a_1a_2a_3\ldots$.

Again, using (4), we get

$$0 \leq a - \widetilde{a}_i = a - \sum_{n=1}^{i} \frac{a_n}{3^n} \leq \frac{1}{3^i} \to 0 \text{ as } i \to \infty.$$

Hence $a = \lim_{i\to\infty} \tilde{a}_i$, or in other words, $a = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$.

(ii) If $a_i \in \{0,1,2\}$, then

$$0.a_1a_2a_3\ldots \leq \sum_{n=1}^{\infty} \frac{2}{3^n} = 1.$$

(iii) We have $1/3 = 0.1000\ldots = 0.0222\ldots$.

(iv) Let $a \in [0,1]$ and assume that a admits two tenary expansions, i.e.

$$a = 0.a_1a_2\cdots \;=\; 0.b_1b_2\cdots\,.$$

Assume that the expansions differ, i.e that there exists $k \geq 1$ such that

$$a_i = b_i \text{ for all } i \leq k \text{ and } a_{k+1} \neq b_{k+1}.$$

Without loss of generality, we may assume that $a_{k+1} < b_{k+1}$. If we show that the only possibility is

$$\begin{cases} b_{k+1} = a_{k+1} + 1; \\ a_i = 2 \quad \text{and} \quad b_i = 0, \quad \forall\, i \geq k+2\,, \end{cases} \tag{5}$$

we can conclude that the proposed identification makes the ternary expansion unique. To this end, observe that for a sequence $c_i \in \{0,1,2\}$, we have

$$\sum_{i=k+2}^{\infty} \frac{c_i}{3^i} \leq \frac{1}{3^{k+1}} \quad \text{with equality if and only if } c_i = 2 \text{ for all } i \geq k+2. \tag{6}$$

Now show (5) by contradiction:

- if $a_{k+1} = 0$ and $b_{k+1} = 2$, then using (6)

$$0 = (0.b_1 \cdots) - (0.a_1 \cdots) \geq \frac{2}{3^{k+1}} - \sum_{i=k+2}^{\infty} \frac{a_i}{3^i} \geq \frac{2}{3^{k+1}} - \frac{1}{3^{k+1}} > 0,$$

  contradiction.

- if $a_{k+1} + 1 = b_{k+1}$ and there is $i \geq k+2$ such that $a_i \neq 2$ or $b_i \neq 0$, then by (6)

$$0 = (0.b_1 \cdots) - (0.a_1 \cdots) = \frac{1}{3^{k+1}} - \sum_{i=k+2}^{\infty} \frac{a_i - b_i}{3^i} > \frac{1}{3^{k+1}} - \frac{1}{3^{k+1}} = 0,$$

  contradiction.