# Serie 1
## Analysis IV, Spring semester
## EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am, via moodle (with the exception of the first exercise which can be handed in until Monday March 6). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

**Exercise 1.** We prove some identities from set theory.

(i) Let $f : X \to Y$ be a function. Show that

$$f^{-1}(A^c) = (f^{-1}(A))^c, \quad f^{-1}(Y) = X, \quad f^{-1}(\emptyset) = \emptyset,$$

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i), \quad f^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(A_i),$$

$$\big[A \cap (B \cup C)\big] \cap B = A \cap B, \quad \big[A \cap (B \cup C)\big] \cap B^c = A \cap B^c \cap C,$$

$$B \subset A \quad \Rightarrow \quad B = A \setminus (A \setminus B)\,,$$

$$\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} [A_i \setminus B_i]\,, \quad \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left[\bigcup_{k=1}^{i} A_k \setminus \bigcup_{k=1}^{i-1} A_k\right].$$

(ii) Prove that if $A \subset \mathbb{R}^n$ and $E_1, E_2, ... \subset \mathbb{R}^n$ a family of disjoint sets, then

$$A \cap \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} [A \cap E_j]\,.$$

**Exercise 2.** Let $J_n = ]c_n, d_n[$ be open intervals covering an the closed interval $[a, b]$, that is

$$]a, b[ \subset [a, b] \subset \bigcup_{n=1}^{N} J_n\,.$$

Show that

$$b - a \leq \sum_{n=1}^{N} \text{length}\,(J_n)\,.$$

*Hints:*

- Assume without loss of generality (and justify it) that no $J_n$ is contained in another $J_m$ and that $J_n \cap [a,b] \neq \emptyset$.
- Order the $c_n$. Deduce an order for $d_n$ and show that $c_{n+1} \leq d_n \quad \forall\, n = 1, \cdots, N-1$.

**Exercise 3.** We show that open sets in $\mathbb{R}^n$ are countable unions of open boxes which can be asked to be disjoint if $n = 1$.

(i) Show that if

$$\{I_\alpha\}_{\alpha \in A}$$

is a family of open intervals, then there is a countable subfamily $\{I_k\}_{k=1}^\infty$ such that

$$\bigcup_{k=1}^\infty I_k = \bigcup_{\alpha \in A} I_\alpha.$$

(ii) Let $\Omega \subset \mathbb{R}$ be an open set. Show that $\Omega$ can be written as a countable union of open disjoint intervals.

(iii) Prove that every open set $\Omega \subset \mathbb{R}^n$ can be written as a countable union of open boxes.

*Hints*:

- For (i) use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. Prove that every $x \in \bigcup_{\alpha \in A} I_\alpha$ belongs to an interval with rational extrema which is contained in some $I_\alpha$.
- In order to prove (ii), define the equivalence relation (and show that it is an equivalence relation) $\sim$ given by

  $$a \sim b \quad \Longleftrightarrow \quad [a,b] \subset \Omega \quad \text{or} \quad [b,a] \subset \Omega,$$

  and then use (i).
- For (iii) recall that since $\Omega$ is open, $\forall x \in \Omega$, $\exists r_x > 0$ such that $B(x, r_x) \subset \Omega$. In addition, prove and use the following claim:

  *Claim:* For any ball $B(x,r)$ there exists a rational box (i.e a box $\prod_{i=1}^n (a_i, b_i)$ with $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{Q}$) contained in $B(x,r)$ and containing $x$.

We provide here a simple example of a sequence of functions which does not converge uniformly, but nonetheless their integral converges to the integral of the pointwise limit. Lebesgue integration will cover such (and much more general) situations with a non-uniform convergence of integrands.

**Exercise 4.** Let $f_n \colon ]0,1[ \to \mathbb{R}$ defined by $f_n(x) = x^n$. Prove that for any compact set $K \subset ]0,1[$

$$f_n \text{ converges uniformly on } ]0,1[ \cap K$$

but $f_n$ does not converge uniformly on $]0,1[$. Show that $\int_0^1 f_n(x)\, dx$ converges to the integral of the pointwise limit of $f_n$ as $n \to \infty$.
*Hint:* Show that for any compact set $K \subset ]0,1[$, there is $\delta > 0$ such that $x \leq 1 - \delta$.

**Exercise 5.** Let $f \in C^1(\mathbb{R}^d)$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. Show that all the following statements are equivalent.

(i) $f$ is convex.

(ii) For any $x, y \in \mathbb{R}^d$,
$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle.$$

(iii) For any $x, y \in \mathbb{R}$,
$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0.$$

If, in addition, $f \in C^2(\mathbb{R}^d)$, then (i)-(iii) are equivalent to

(iv) For any $x, v \in \mathbb{R}^d$,
$$\langle \nabla^2 f(x) v, v \rangle \geq 0.$$
where $\nabla^2 f(x)$ denotes the Hessian matrix $\nabla^2 f(x) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial^2 x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_d \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial^2 x_d} \end{pmatrix}$

*Hint:* For this last part, recall that for $g \in C^1$
$$\int_{t_0}^{t_1} g'(t)\, dt = g(t_1) - g(t_0).$$