Serie 1

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am, via moodle (with the exception of the first exercise which can be handed in until Monday March 6). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. We prove some identities from set theory.

(i) Let $f : X \to Y$ be a function. Show that

$$f^{-1}(A^c) = (f^{-1}(A))^c, \quad f^{-1}(Y) = X, \quad f^{-1}(\emptyset) = \emptyset,$$

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i), \quad f^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(A_i),$$

$$\big[A \cap (B \cup C)\big] \cap B = A \cap B, \quad \big[A \cap (B \cup C)\big] \cap B^c = A \cap B^c \cap C,$$

$$B \subset A \quad \Rightarrow \quad B = A \setminus (A \setminus B),$$

$$\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} [A_i \setminus B_i], \quad \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left[\bigcup_{k=1}^{i} A_k \setminus \bigcup_{k=1}^{i-1} A_k\right].$$

(ii) Prove that if $A \subset \mathbb{R}^n$ and $E_1, E_2, ... \subset \mathbb{R}^n$ a family of disjoint sets, then

$$A \cap \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} [A \cap E_j].$$

Solution:

(i) The eight first statements are trivial. Recall that if $f\colon X \to Y$, then for any $E \subset Y$,

$$f^{-1}(E) := \{x \in X : f(x) \in E\}.$$

We prove the second last statement:

$$
\begin{aligned}
x \in \bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} B_i \quad &\Rightarrow \quad \exists i \in \mathbb{N} \text{ such that } x \in A_i \text{ and } x \notin B_j \text{ for all } j \in \mathbb{N} \\
&\Rightarrow \quad \exists i \in \mathbb{N} \text{ such that } x \in A_i \setminus B_i \\
&\Rightarrow \quad x \in \bigcup_{i=1}^{\infty} [A_i \setminus B_i] \,.
\end{aligned}
$$

Now, for the last statement, it is clear that we have the inclusion

$$
\bigcup_{i=1}^{\infty} A_i \supset \bigcup_{i=1}^{\infty} \left[\bigcup_{k=1}^{i} A_k \setminus \bigcup_{k=1}^{i-1} A_k \right].
$$

We need to show the reverse inclusion. Note that

$$
\bigcup_{i=1}^{\infty} \left[\bigcup_{k=1}^{i} A_k \setminus \bigcup_{k=1}^{i-1} A_k \right] \supset \bigcup_{k=1}^{1} A_k \setminus \bigcup_{k=1}^{0} A_k = A_1 \setminus \emptyset = A_1.
$$

Let now $x \in \bigcup_{i=1}^{\infty} A_i$. If $x \in A_1$, then from the inclusion above

$$
x \in \bigcup_{i=1}^{\infty} \left[\bigcup_{k=1}^{i} A_k \setminus \bigcup_{k=1}^{i-1} A_k \right].
$$

More generally, if $x \in [\bigcup_{i=1}^{\infty} A_i] \setminus A_1$, take the smallest $j \geq 2$ such that $x \in A_j$ (then $x \notin A_k$, for any $1 \leq k < j$). It follows that

$$
x \in A_j \setminus \bigcup_{k=1}^{j-1} A_k \subset \bigcup_{i=1}^{\infty} \left[\bigcup_{k=1}^{i} A_k \setminus \bigcup_{k=1}^{i-1} A_k \right].
$$

(ii) We have the following equivalences which yield the final result:

$$
\begin{aligned}
x \in A \cap \bigcup_{j=1}^{\infty} E_j \quad &\Leftrightarrow \quad x \in A \text{ and } x \in \bigcup_{j=1}^{\infty} E_j \\
&\Leftrightarrow \quad x \in A \text{ and } \exists i \in \mathbb{N} \text{ such that } x \in E_i \\
&\Leftrightarrow \quad \exists i \in \mathbb{N} \text{ such that } x \in A \cap E_i \\
&\Leftrightarrow \quad x \in \bigcup_{j=1}^{\infty} A \cap E_j.
\end{aligned}
$$

Exercise 2. Let $J_n =]c_n, d_n[$ be open intervals covering an the closed interval $[a,b]$, that is

$$
]a,b[\subset [a,b] \subset \bigcup_{n=1}^{N} J_n \,.
$$

Show that

$$
b - a \leq \sum_{n=1}^{N} \text{length}\,(J_n) \,.
$$

Hints:

- Assume without loss of generality (and justify it) that no J_n is contained in another J_m and that $J_n \cap [a, b] \neq \emptyset$.
- Order the c_n. Deduce an order for d_n and show that $c_{n+1} \leq d_n \quad \forall n = 1, \cdots, N-1$.

Solution: The solution is divided into 4 steps.

Step 1: We show that without loss of generality we can assume that no J_n is contained in another J_m and that $J_n \cap [a, b] \neq \emptyset$ for all $n = 1, \ldots, N$.

Indeed, assume that the statement is true under these additional assumptions and then consider a case where $J_n \subset J_m$. Eliminate J_n and continue this process of eliminating intervals until no interval is contained in another. By the assumption above, the statement is true in this modified case and it is easily seen that it must therefore be true even in the initial more general case. The same argument of eliminating intervals applies for the assumption that $J_n \cap [a, b] \neq \emptyset$.

Step 2: We order the c_n and deduce an order of the d_n.

Order the c_n such that $c_1 < \cdots < c_n < c_{n+1} < \cdots < c_N$. Note that then

$$d_1 < \cdots < d_n < d_{n+1} < \cdots < d_N .$$

Indeed if $d_n \leq d_k$ for some $k < n$, then

$$c_k < c_n < d_n \leq d_k$$

and thus $J_n \subset J_k$, which contradicts our assumption.

Step 3: We show the inequality $c_{n+1} < d_n$.

By contraction, assume that for some n we have $c_n < d_n \leq c_{n+1} < d_{n+1}$. This implies that

$$\bigcup_{n=1}^{N} J_n \cap \,]d_n, c_{n+1}[\, = \emptyset.$$

However $[a, b] \subset \bigcup_{n=1}^{N} J_n$, and therefore $[a, b] \cap \,]d_n, c_{n+1}[= \emptyset$. Then either

$$a < b < d_n \leq c_{n+1} < d_{n+1} \quad \Rightarrow \quad J_{n+1} \cap [a, b] = \emptyset$$

which is a contradiction, or

$$c_n < d_n \leq c_{n+1} < a < b \quad \Rightarrow \quad J_n \cap [a, b] = \emptyset$$

which is also a contradiction.

Step 4: Conclusion.

We conclude

$$\sum_{n=1}^{N} \text{length}\,(J_n) = \sum_{n=1}^{N} (d_n - c_n) \geq \sum_{n=1}^{N} (d_n - c_n) - \sum_{n=1}^{N-1} (d_n - c_{n+1}) = d_N - c_1 \geq b - a.$$

Exercise 3. We show that open sets in $\mathbb{R}^n$ are countable unions of open boxes which can be asked to be disjoint if $n = 1$.

(i) Show that if

$$\{I_\alpha\}_{\alpha \in A}$$

is a family of open intervals, then there is a countable subfamily $\{I_k\}_{k=1}^{\infty}$ such that

$$\bigcup_{k=1}^{\infty} I_k = \bigcup_{\alpha \in A} I_\alpha.$$

(ii) Let $\Omega \subset \mathbb{R}$ be an open set. Show that Ω can be written as a countable union of open disjoint intervals.

(iii) Prove that every open set $\Omega \subset \mathbb{R}^n$ can be written as a countable union of open boxes.

Hints:

- For (i) use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. Prove that every $x \in \bigcup_{\alpha \in A} I_\alpha$ belongs to an interval with rational extrema which is contained in some I_α.
- In order to prove (ii), define the equivalence relation (and show that it is an equivalence relation) $\sim$ given by

$$a \sim b \quad \Longleftrightarrow \quad [a, b] \subset \Omega \quad \text{or} \quad [b, a] \subset \Omega,$$

  and then use (i).
- For (iii) recall that since Ω is open, $\forall x \in \Omega$, $\exists r_x > 0$ such that $B(x, r_x) \subset \Omega$. In addition, prove and use the following claim:

  Claim: For any ball $B(x, r)$ there exists a rational box (i.e a box $\prod_{i=1}^n (a_i, b_i)$ with $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{Q}$) contained in $B(x, r)$ and containing x.

Solution:

(i) Put

$$B := \bigcup_{\alpha \in A} I_\alpha.$$

For any $x \in B$, define the set

$$\mathcal{I}_x = \{(a, b, \alpha) \in \mathbb{Q} \times \mathbb{Q} \times A \mid a < x < b \text{ and }]a, b[\subset I_\alpha\}.$$

Then, for all $x \in B$, $\mathcal{I}_x$ is non-empty. Indeed, if $x \in B$, then there is $\alpha \in A$ such that $x \in I_\alpha$. Moreover, since I_α is open, there is $\varepsilon > 0$ such that

$$x \in]x - \varepsilon, x + \varepsilon[\subset I_\alpha.$$

By density of $\mathbb{Q}$ in $\mathbb{R}$, there is $a \in]x - \varepsilon, x[\cap \mathbb{Q}$ and $b \in]x, x + \varepsilon[\cap \mathbb{Q}$. We get $(a, b, \alpha) \in \mathcal{I}_x$ which is therefore non-empty. Then applying the axiom of choice to the family $\{\mathcal{I}_x\}_{x \in B}$, we get for all $x \in B$, $a_x, b_x \in \mathbb{Q}$ and $\alpha_x \in A$ such that $(a_x, b_x, \alpha_x) \in \mathcal{I}_x$, i.e.,

$$x \in J_x :=]a_x, b_x[\subset I_{\alpha_x}$$

By construction, we have

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{x \in B} J_x.$$

The family $\{J_x\}_{x \in B}$ is countable since it is a subset of the family of intervals with rational extrema which is countable (because there is a bijection with $\mathbb{Q} \times \mathbb{Q}$ which is countable). Thus, there is a family $\{J_{x_i}\}_{i=1}^{\infty} \subset \{J_x\}_{x \in B}$ such that

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} J_{x_i}.$$

It follows that

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} J_{x_i} \subset \bigcup_{i=1}^{\infty} I_{\alpha_{x_i}} \subset \bigcup_{\alpha \in A} I_\alpha,$$

hence the result.

(ii) We define the following equivalence relation $\sim$:

$$a \sim b \quad \iff \quad [a, b] \subset \Omega \quad \text{or} \quad [b, a] \subset \Omega.$$

It is easily shown that it is an equivalence relation, i.e. it is reflexive, symmetric and transitive. We show that the equivalence classes form partition of Ω in disjoint (this is obvious) open intervals. Indeed, let $a \in \Omega$, and $C(a)$ be its equivalence class. Let $x, y \in C(a)$, implying that $x \sim y$, which means that $[x, y] \subset \Omega$ or $[y, x] \subset \Omega$. As a consequence, $C(a)$ is convex and therefore an interval. To show that $C(a)$ is open, let again $x \in C(a) \subset \Omega$. Since Ω is open there is ε such that $]x - \varepsilon, x + \varepsilon[\subset \Omega$ and by transitivity it is clear that any $y \in]x - \varepsilon, x + \varepsilon[$ satisfies $y \in C(a)$. Finally, $]x - \varepsilon, x + \varepsilon[\subset C(a)$ proving that $C(a)$ is open. Until now, we have proved that $\sim$ partions Ω into equivalence classes that are open intervals so that we get a family of open disjoint intervals $\{I_\alpha\}_{\alpha \in A}$ such that

$$\Omega = \bigcup_{\alpha \in A} I_\alpha.$$

Now using (i), there is a countable subfamily $\{I_k\}_{k=1}^{\infty}$ such that

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{k=1}^{\infty} I_k.$$

But since the intervals are all disjoint, this implies $\{I_\alpha\}_{\alpha \in A} = \{I_k\}_{k=1}^{\infty}$, hence $\{I_\alpha\}_{\alpha \in A}$ is countable which finishes the proof.

(iii) As mentioned in the exercise, a box $\prod_{i=1}^{n} (a_i, b_i)$ is called rational if all the components a_i, b_i are rational numbers. Notice that the family of all rational boxes is countable (there is a bijection between this family and $\mathbb{Q}^{2n}$ which is countable). We first prove the claim. For each $1 \leq i \leq n$, let a_i, b_i be rational numbers such that

$$x_i - \frac{r}{n} < a_i < x_i < b_i < x_i + \frac{r}{n}.$$

Then it is clear that the box $\prod_{i=1}^n (a_i, b_i)$ is rational and contains x. The triangular inequality allows us to prove that this box is contained in $B(x, r)$. This proves the claim. Finally, let $\Omega \subset \mathbb{R}^n$ and let Σ be the family of rational cubes contained in Ω. We will prove

$$\Omega = \bigcup_{B \in \Sigma} B,$$

which gives the desired result. The fact that the union over Σ is contained in Ω is obvious because every element in Σ is contained in Ω. On the other hand, since Ω is open, for any $x \in \Omega$ there is $r_x > 0$ such that $B(x, r_x) \subset \Omega$. Then, by the claim, there is rational a box B such that $x \in B \subset B(x, r_x)$. In particular, $B \in \Sigma$ and thus we get $x \in \cup_{B \in \Sigma} B$. Since this is true for any $x \in \Omega$, Ω is contained in the union over Σ. Hence, we deduce the desired equality. Note that Σ is countable since it is a subset of the family of all rational boxes, which is countable.

We provide here a simple example of a sequence of functions which does not converge uniformly, but nonetheless their integral converges to the integral of the pointwise limit. Lebesgue integration will cover such (and much more general) situations with a non-uniform convergence of integrands.

Exercise 4. Let $f_n \colon]0,1[\to \mathbb{R}$ defined by $f_n(x) = x^n$. Prove that for any compact set $K \subset]0,1[$

$$f_n \text{ converges uniformly on }]0,1[\cap K$$

but f_n does not converge uniformly on $]0,1[$. Show that $\int_0^1 f_n(x)\,dx$ converges to the integral of the pointwise limit of f_n as $n \to \infty$.
Hint: Show that for any compact set $K \subset]0,1[$, there is $\delta > 0$ such that $x \leq 1 - \delta$.

Solution: Let $K \subset]0,1[$ be a compact set. Then, since K is compact and $\mathbb{R} \setminus]0,1[$ is closed, with empty intersection, there is

$$\gamma = \min_{x \in K, y \in \mathbb{R} \setminus]0,1[} |x - y| > 0.$$

Thus, by defining $\delta = \gamma/2$, we have $x < 1 - \delta$ for any $x \in K$. As a consequence, we get $K \subset]0, 1-\delta[$. We know that $(1-\delta)^n \to 0$ and

$$\sup_{x \in K} |x^n| \leq \sup_{0 \leq x \leq 1-\delta} x^n = (1-\delta)^n.$$

Thus, for any $\varepsilon > 0$, there exists N sufficiently large such that for any $n \geq N$, we have

$$\sup_{x \in K} |x^n| \leq \varepsilon.$$

This proves uniform convergence on K. From this, we deduce the local uniform convergence.

Next, we prove that f_n does not converge uniformly on $]0,1[$. Indeed, $f_n(x) \to 0$ as $n \to \infty$, for any $x \in]0,1[$. Assume for a contradiction that $f_n \to 0$ uniformly on $]0,1[$. Then for any $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$\sup_{x \in]0,1[} |f_n| \leq \varepsilon.$$

However,

$$\sup_{x \in]0,1[} |f_n| = \sup_{x \in]0,1[} |x^n| = 1,$$

which gives the desired contradiction. Finally,

$$\int_0^1 f_n(x)\, dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$$

such that $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0$ and since $f_n \to 0$ pointwise, this shows the second claim.

Exercise 5. Let $f \in C^1(\mathbb{R}^d)$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. Show that all the following statements are equivalent.

(i) f is convex.

(ii) For any $x, y \in \mathbb{R}^d$,

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle.$$

(iii) For any $x, y \in \mathbb{R}$,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0.$$

If, in addition, $f \in C^2(\mathbb{R}^d)$, then (i)-(iii) are equivalent to

(iv) For any $x, v \in \mathbb{R}^d$,

$$\langle \nabla^2 f(x) v, v \rangle \geq 0.$$

where $\nabla^2 f(x)$ denotes the Hessian matrix $\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 f}{\partial^2 x_d} \end{pmatrix}$

Hint: For this last part, recall that for $g \in C^1$

$$\int_{t_0}^{t_1} g'(t)\, dt = g(t_1) - g(t_0).$$

Solution: We will prove: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\xLeftrightarrow{f \in C^2(\mathbb{R}^d)}$ (iv).

We begin with (i) $\Rightarrow$ (ii): For any $\lambda \in]0,1[$ and every $x, y \in \mathbb{R}^d$ by convexity of f,

$$\lambda(f(x) - f(y)) + f(y) = \lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y),$$

which implies,

$$f(x) - f(y) \geq \frac{1}{\lambda}[f(\lambda x + (1-\lambda)y) - f(y)].$$

Letting $\lambda \to 0$ (observe: $\lambda x + (1-\lambda)y = y + \lambda(x-y)$), we get,

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle.$$

Now for (ii) $\Rightarrow$ (i): We have using (ii) that

$$f(x) \geq f(\lambda x + (1-\lambda)y) + (1-\lambda)\langle \nabla f(\lambda x + (1-\lambda)y), x - y\rangle \quad (1)$$

$$f(y) \geq f(\lambda x + (1-\lambda)y) - \lambda\langle \nabla f(\lambda x + (1-\lambda)y), x - y\rangle. \quad (2)$$

Multiplying (1) by λ, (2) by $(1-\lambda)$ and summing them, yield convexity.

Then for (ii) $\Rightarrow$ (iii): We have using (ii) that

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle \quad (3)$$

$$f(y) \geq f(x) - \langle \nabla f(x), x - y\rangle. \quad (4)$$

Summing (3) and (4) yields the desired result.

In addition, for (iii) $\Rightarrow$ (ii): Let $\lambda \in]0,1[$, $x, y \in \mathbb{R}^d$ and define

$$z = \frac{x-y}{\lambda} + y$$

and a function $\phi\colon \mathbb{R}^d \to \mathbb{R}$ by $\phi(\mu) = f(y + \mu(z-y))$. Note that $\phi(\lambda) = f(x)$ and $\phi(0) = f(y)$. In addition,

$$\phi'(\mu) = \langle \nabla f(y + \mu(z-y)), z - y\rangle$$

and therefore by (iii)

$$\phi'(\mu) - \phi'(0) = \frac{1}{\mu}\langle \nabla f(y + \mu(z-y)) - \nabla f(y), (y + \mu(z-y)) - y\rangle \geq 0.$$

Integrating this ineqality from 0 to λ gives

$$\int_0^\lambda (\phi'(\mu) - \phi'(0))\, d\mu = \phi(\lambda) - \phi(0) - \lambda\phi'(0) \geq 0,$$

which implies, using $\lambda(z-y) = x - y$,

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle.$$

From now on, assume $f \in C^2(\mathbb{R}^d)$. We show (iii) $\Rightarrow$ (iv): Indeed, by (iii) it holds for every $v \in \mathbb{R}^d$

$$\left\langle \frac{\nabla f(x + \varepsilon v) - \nabla f(x)}{\varepsilon}, v \right\rangle \geq 0.$$

Letting $\varepsilon \to 0$, gives $\langle \nabla^2 f(x)v, v\rangle \geq 0$.

Finally (iv) $\Rightarrow$ (iii): Use the hint with the function $g(t) = \langle \nabla f(y + t(x - y)), x - y\rangle$, $t_0 = 0$ and $t_1 = 1$. This yields

$$\begin{aligned}\langle \nabla f(x) - \nabla f(y), x - y\rangle &= \int_0^1 \left\langle \frac{d}{dt}\nabla f(y + t(x - y)), x - y\right\rangle\, dt\\ &= \int_0^1 \left\langle \nabla^2 f(y + t(x - y))(x - y), x - y\right\rangle\, dt \geq 0\end{aligned}$$

due to (iv).