Serie 13

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Use the Fourier transform to solve the following initial value problem

$$\begin{cases} \Delta u = 0 & (x,y) \in \Omega, \\ u(x,0) = \dfrac{8x^2}{(1+x^2)^2} & x \in \mathbb{R}, \\ \lim_{y\to+\infty} u(x,y) = 0 & x \in \mathbb{R}\,. \end{cases}$$

Hint: Use that for $\omega \neq 0$ we have

$$\begin{aligned} f(x) &= \frac{1}{x^2+\omega^2} \quad &\Rightarrow \quad \hat{f}(\xi) &= \pi \frac{e^{-2\pi|\xi\omega|}}{|\omega|}, \\ g(x) &= \frac{4x^2}{(x^2+\omega^2)^2} \quad &\Rightarrow \quad \hat{g}(\xi) &= 2\pi(\frac{1}{|\omega|} - 2\pi|\xi|)e^{-2\pi|\xi\omega|}. \end{aligned}$$

Solution: We begin with a formal derivation of the solution. In order to solve the PDE we will take the Fourier transform with respect to x, i.e.

$$\hat{u}(\xi,y) = \mathcal{F}_x(u)(\xi,y) = \int_{-\infty}^{+\infty} u(x,y)\mathrm{e}^{-2\pi i x\xi}\,dx.$$

Note that

$$\mathcal{F}_x(\partial_{xx}u)(\xi,y) = (2\pi i\xi)^2\hat{u}(\xi,y) \quad \text{and} \quad \mathcal{F}_x(\partial_{yy}u)(\xi,y) = \partial_{yy}\hat{u}(\xi,y). \tag{1}$$

Observe that formally, if we are allowed to exchange limit and integration by, say, dominated convergence, then the third equation can be rewritten as

$$\lim_{y\to\infty} \hat{u}(\xi,y) = \lim_{y\to\infty} \int_{-\infty}^{+\infty} u(x,y)\mathrm{e}^{-2\pi i x\xi}\,dx = 0 \tag{2}$$

for all $\xi \in \mathbb{R}$. Using (1)-(2), the PDE can now be rewritten as

$$\begin{cases} -(2\pi\xi)^2\hat{u}(\xi,y) + \partial_{yy}\hat{u}(\xi,y) = 0 & (\xi,y) \in \Omega, \\ \hat{u}(\xi,0) = \hat{h}(\xi) & \xi \in \mathbb{R}, \\ \lim_{y\to+\infty} \hat{u}(\xi,y) = 0 & \xi \in \mathbb{R}. \end{cases} \tag{3}$$

The first two equations are an ODE with respect to y and the solution is given by

$$\begin{aligned} \hat{u}(\xi,y) &= \hat{h}(\xi)\cosh(2\pi|\xi|y) + \frac{\gamma}{2\pi|\xi|}\sinh(2\pi|\xi|y) \\ &= \hat{h}(\xi)\frac{\mathrm{e}^{2\pi|\xi|y} + \mathrm{e}^{-2\pi|\xi|y}}{2} + \frac{\gamma}{2\pi|\xi|}\frac{\mathrm{e}^{2\pi|\xi|y} - \mathrm{e}^{-2\pi|\xi|y}}{2} \\ &= \left(\hat{h}(\xi) + \frac{\gamma}{2\pi|\xi|}\right)\frac{\mathrm{e}^{2\pi|\xi|y}}{2} + \left(\hat{h}(\xi) - \frac{\gamma}{2\pi|\xi|}\right)\frac{\mathrm{e}^{-2\pi|\xi|y}}{2} \end{aligned}$$

where γ does not depend on y. We need to determine the value of γ such that the third equation is satisfied; that is, we need to choose γ such that $\lim_{y\to+\infty} \hat{u}(\xi,y) = 0$ for all $\xi \in \mathbb{R}$. Looking at the equation just above, it follows that

$$\gamma = -2\pi|\xi|\hat{h}(\xi).$$

We get

$$\hat{u}(\xi,y) = \hat{h}(\xi)\mathrm{e}^{-2\pi|\xi|y}.$$

Now, in order to recover u, we compute the Fourier inversion. We observe that $\hat{h}(\xi) = 4\pi(1 - 2\pi|\xi|)e^{-2\pi|\xi|}$ by the hint and hence

$$\begin{aligned} \hat{u}(\xi,y) &= 4\pi(1-2\pi|\xi|)e^{-2\pi|\xi|(y+1)} = 4\pi\left(\frac{1}{1+y} - 2\pi|\xi|\right)e^{-2\pi|\xi|(y+1)} + 4\pi\left(1 - \frac{1}{1+y}\right)e^{-2\pi|\xi|(y+1)} \\ &= 2\,\mathcal{F}_x\left(\frac{4x^2}{(x^2+(1+y)^2)^2}\right)(\xi) + 4y\,\mathcal{F}_x\left(\frac{1}{x^2+(1+y)^2}\right)(\xi), \end{aligned}$$

where we again used the two hints with $\omega = 1 + y$. Hence, taking the Fourier inversion, we formally obtain that

$$\begin{aligned} u(x,y) = \mathcal{F}_x^{-1}(\hat{u})(x,y) = \int_{-\infty}^{+\infty} \hat{h}(\xi)\mathrm{e}^{-2\pi|\xi|y}\mathrm{e}^{2\pi ix\xi}\,d\xi &= \frac{8x^2}{(x^2+(1+y)^2)^2} + \frac{4y}{(1+y)^2+x^2} \\ &= 4\frac{y(1+y)^2 + x^2(2+y)}{(x^2+(1+y)^2)^2}. \end{aligned}$$

From this expression, it is now straightforward to verify that $u \in C^2(\Omega)$ satisfies all three equations of the PDE.

Exercise 2.

(i) Let $f \in L^1(0, 2\pi)$. Find, formally, the solution $u = u(x,t)$ of the initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & (x,t) \in (0,2\pi) \times (0,\infty), \\ u(0,t) = u(2\pi, t) & t > 0, \\ u_x(0,t) = u_x(2\pi,t) & t > 0, \\ u(x,0) = f(x) & x \in (0, 2\pi)\,. \end{cases}$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic. Discuss under which additional assumptions on f , we have that

$$\lim_{t \to 0} u(x,t) = f(x) \quad \text{uniformly in } x.$$

Solution:

(i) We use separation of variables and make the ansatz that u is in the form $u(x,t) = \varphi(x)\psi(t)$. The first equation then becomes

$$\frac{\phi'(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda \quad \forall (x,t) \in (0,2\pi) \times (0,\infty)$$

for a suitable $\lambda \in \mathbb{R}$. We then obtain the two ODEs:

$$\begin{cases} \varphi''(x) - \lambda\varphi(x) = 0 & x \in (0,2\pi) \\ \varphi(0) = \varphi(2\pi), \\ \varphi'(0) = \varphi'(2\pi), \end{cases} \tag{4}$$

and

$$\psi'(t) - \lambda\psi(t) = 0 \quad t > 0\,. \tag{5}$$

We know that the problem (4) admits non-trivial solutions only if $\lambda = -n^2$ for $n \in \mathbb{N}_{\geq 0}$ and the solutions are given by

$$\varphi_n(x) = \begin{cases} \frac{a_0}{2} & \text{if } n = 0, \\ a_n \cos(nx) + b_n \sin(nx) & \text{otherwise} \end{cases},$$

for some real coefficients $\{a_n\}$. For $\lambda = -n^2$ the solution to (5) is given by

$$\psi_n(t) = c_n \mathrm{e}^{-n^2 t}.$$

Thus, since any superposition of solutions is also a solution, the general solution to the two ODE's is, formally, of the form

$$u(x,t) = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] c_n \mathrm{e}^{-n^2 t}.$$

For simplicity, we rewrite it as

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] e^{-n^2 t}.$$

Now, in order to determine the coefficients a_n and b_n we use the initial condition. Indeed, formally

$$a_k = \frac{1}{\pi} \int_0^{2\pi} u(x,0) \cos(kx)\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx)\, dx, \qquad \text{for all } k \geq 0\,.$$

Similarly we find

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx)\, dx \qquad \text{for all } k \geq 1\,.$$

With these choices for the coefficients a_n and b_n, u formally solves the initial value problem.

(ii) If the Fourier series of f converges uniformly to f, we have that pointwise

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

and hence,

$$u(x,t) - f(x) = \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)](e^{-n^2 t} - 1)\,. \tag{6}$$

Solution 1: $f \in C^4(\mathbb{R})$.
The assumption guarantees uniform convergence of the Fourier series and moreover, there is a constant such that

$$|a_n|, |b_n| \leq \frac{\gamma}{n^4}\,.$$

We can estimate therefore

$$|u(x,t) - f(x)| = \sum_{n=1}^{\infty} [|a_n| + |b_n|](1 - e^{-n^2 t}) \leq \sum_{n=1}^{\infty} \frac{2\gamma}{n^4} n^2 t = 2\gamma t \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{<\infty}\,.$$

We deduce that $|u(x,t) - f(x)| \to 0$, uniformly in x as $t \to \infty$, which proves that

$$\lim_{t \to 0} u(x,t) = f(x) \quad \text{uniformly in } x.$$

Solution 2: $f \in C^{1,\alpha}(\mathbb{R})$.
The assumption $f \in C^{1,\alpha}(\mathbb{R})$ (i.e. $f \in C^1(\mathbb{R})$ and $f' \in C^{\alpha}(\mathbb{R})$) guarantees uniform convergence of the Fourier series and moreover, there is a constant such that

$$|a_n|, |b_n| \leq \frac{\gamma}{n^{1+\alpha}}\,. \tag{7}$$

To show the uniform convergence, we fix $\varepsilon > 0$. We estimate using (6), (7) and the triangular inequality, for a $N \geq 1$ yet to be chosen,

$$|u(x,t) - f(x)| \leq 2\gamma(1 - e^{-N^2 t}) \sum_{n=1}^{N} \frac{1}{n^{1+\alpha}} + 4\gamma \sum_{n=N}^{\infty} \frac{1}{n^{1+\alpha}} . \tag{8}$$

We now first choose $N = N(\varepsilon) \geq 1$ large enough such that $4\gamma \sum_{n=N}^{\infty} n^{-(1+\alpha)} \leq \frac{\varepsilon}{2}$. Second, we choose $t_0 = t_0(N) > 0$ small enough, such that for all $0 < t < t_0$, it holds that

$$2\gamma(1 - e^{-N^2 t}) \sum_{n=1}^{N} \frac{1}{n^{1+\alpha}} < \frac{\varepsilon}{2} .$$

Hence, for all $0 < t < t_0$ and for all $x \in \mathbb{R}$, we have $|u(x,t) - f(x)| \leq \varepsilon$, showing the claimed uniform convergence.

Exercise 3. Let $f \in L^1(\mathbb{R})$ and assume that $f(x) = \overline{f(-x)}$ for all $x \in \mathbb{R}$, $\hat{f} \geq 0$ pointwise, and f is continuous in a neighbourhood of 0.

(i) Show that $\hat{f} \in L^1(\mathbb{R})$.

(ii) Give an example showing that $\hat{f}$ doesn't need to be in $L^1(\mathbb{R})$ if we drop the assumption $\hat{f} \geq 0$.

Hint: Choose $\varphi \in C^\infty(\mathbb{R})$ to be a standard Gaussian and consider $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi\left(\frac{x}{\varepsilon}\right)$. Use and prove that $\lim_{\varepsilon \to 0}(f * \varphi_\varepsilon)(0) = f(0)$.

Solution:

(i) Let φ be a Gaussian, i.e. $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$, with $\|\varphi\|_{L^1(\mathbb{R})} = 1$ and define for $\varepsilon > 0$, the rescaling $\varphi_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right).$$

Note that by the positivity of φ_ε and by a change of variables,

$$\|\varphi_\varepsilon\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \varphi_\varepsilon(x)\,dx = \int_{\mathbb{R}} \varphi(x)\,dx = 1.$$

Since φ is a Gaussian, $\hat{\varphi}(\xi) = (2\pi)^{-1/2} e^{-\xi^2/2}$ is itself a Gaussian and hence $\hat{\varphi} \geq 0$. Notice that for every $\delta > 0$ fixed, we have

$$\lim_{\varepsilon \to 0} \sup_{|x| > \delta} |\varphi_\varepsilon(x)| = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \|\varphi_\varepsilon \mathbb{1}_{[|x|>\delta]}\|_{L^1(\mathbb{R})} = 0. \tag{9}$$

We claim $(f * \varphi_\varepsilon)(0) \to f(0)$ as $\varepsilon \to 0$. Indeed, let $\tau > 0$ be arbitrary. Define $M := |f(0)| + \|f\|_{L^1(\mathbb{R})}$. Since f is continuous in 0, there is $\nu = \nu(\tau) > 0$ such that

$$|f(x) - f(0)| < \frac{\tau}{2} \quad \text{for any} \quad |x| < \nu.$$

Moreover, due to (9), there is $\bar{\varepsilon} = \bar{\varepsilon}(\tau, M, \nu) > 0$ such that for all $0 < \varepsilon \leq \bar{\varepsilon}$,

$$\sup_{|x| > \nu} |\varphi_\varepsilon(x)| < \frac{\tau}{2M} \quad \text{and} \quad \|\varphi_\varepsilon \mathbb{1}_{[|x|>\nu]}\|_{L^1(\mathbb{R})} < \frac{\tau}{2M}.$$

Combining the three estimates and using that $\int \varphi_\varepsilon(y)\,dy = 1$, we have for any $0 < \varepsilon \le \bar\varepsilon$ that

$$\begin{aligned}
|(f*\varphi_\varepsilon)(0) - f(0)| &= \left|\int_{\mathbb{R}} f(-y)\varphi_\varepsilon(y)\,dy - f(0)\right| \\
&= \left|\int_{\mathbb{R}} [f(-y) - f(0)]\varphi_\varepsilon(y)\,dy\right| \\
&\le \int_{|y|<\nu} |f(-y) - f(0)|\varphi_\varepsilon(y)\,dy + \int_{|y|\ge\nu} |f(-y) - f(0)|\varphi_\varepsilon(y)\,dy \\
&\le \frac{\tau}{2}\int_{|y|<\nu} \varphi_\varepsilon(y)\,dy + \int_{|y|\ge\nu} |f(-y)|\varphi_\varepsilon(y)\,dy + |f(0)|\int_{|y|\ge\nu} \varphi_\varepsilon(y)\,dy \\
&\le \frac{\tau}{2} + \frac{\tau}{2M}\left(\int_{|y|\ge\nu} |f(-y)|\,dy + |f(0)|\right) \le \frac{\tau}{2} + \frac{\tau}{2} = \tau
\end{aligned}$$

Since τ was arbitrary, this proves that $(f*\varphi_\varepsilon)(0) \to f(0)$ as $\varepsilon \to 0$.

We now use the Fourier transform: since $f, \varphi_\varepsilon \in L^1(\mathbb{R})$, we can use the properties the Fourier transform with respect to convolution and dilations to obtain that for all $\xi \in \mathbb{R}$

$$\mathcal{F}(f*\varphi_\varepsilon)(\xi) = \hat f(\xi)\hat\varphi_\varepsilon(\xi) = \hat f(\xi)\hat\varphi(\varepsilon\xi) \ge 0.$$

Since $\hat f$ is bounded as Fourier transfom of an L^1-function and $\hat\varphi$ is a Gaussian (in particular $\hat\varphi \in L^1$), we have that $\hat f\hat\varphi_\varepsilon \in L^1(\mathbb{R})$. We can thus apply the Fourier inversion in order to get for every $x \in \mathbb{R}$

$$\mathcal{F}^{-1}(\hat f(\xi)\hat\varphi(\varepsilon\xi))(x) = (f*\varphi_\varepsilon)(x).$$

Evaluating in $x = 0$ gives

$$\int_{\mathbb{R}} \hat f(\xi)\hat\varphi(\varepsilon\xi)\,d\xi = (f*\varphi_\varepsilon)(0).$$

It now follows from Fatou's lemma (using that $\hat\varphi(0) = \int \varphi(y)\,dy = 1$) that

$$||\hat f||_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \hat f(\xi)\,d\xi \le \liminf_{\varepsilon\to0}\int_{\mathbb{R}} \hat f(\xi)\hat\varphi(\varepsilon\xi)\,d\xi = \liminf_{\varepsilon\to0}(f*\varphi_\varepsilon)(0) = f(0),$$

hence $\hat f \in L^1(\mathbb{R})$.

(ii) Just take the indicator function of [-1,1]. Its Fourier transform is $\frac{\sin(2\pi\xi)}{\pi\xi}$ which is not Lebesgue integrable.

Exercise 4. Consider the 1-dimensional wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & (x,t) \in \mathbb{R}\times(0,\infty) \\ u(x,0) = g(x), \partial_t u(x,0) = h(x) & x \in \mathbb{R}\,. \end{cases}$$

with $g, h \in L^2(\mathbb{R})$.

(i) Find, formally, a representation formula for u .

(ii) Assume that $u \in C^2(\mathbb{R} \times (0, \infty))$ such that $u(\cdot, t)$ has compact support in space for every fixed time $t > 0$ and that $g \in C_c^1(\mathbb{R})$ and that $h \in C_c(\mathbb{R})$. Define the total energy at time t, that is the sum of the kinetic energy and the potential energy,

$$E(t) := \frac{1}{2} \int_{\mathbb{R}} (u_t(x,t))^2 + (u_x(x,t))^2 \, dx$$

for $t \geq 0$. Show that $E(t) = E(0)$ for $t > 0$.

(iii) ($\star$) Under the assumptions of (ii), show that asymptotically as $t \to \infty$, the total energy splits equally into its kinetic and potential parts; that is

$$\lim_{t \to \infty} \int (u_x(x,t))^2 \, dx = \lim_{t \to \infty} \int (u_t(x,t))^2 \, dx = E(0).$$

Hint: Show that for every $f \in C_c^\infty(\mathbb{R})$, it holds

$$\lim_{t \to \infty} \int_{\mathbb{R}} \cos(t 2\pi |\xi|) \sin(t 2\pi |\xi|) f(\xi) \, d\xi = 0 \,. \tag{10}$$

Solution:

(i) We have already seen in class that

$$u(x,t) = \frac{1}{2}\Big(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y) dy \Big).$$

Let's use a different technique than the one from class to derive this formula.

We begin with a formal derivation of the representation formula. In order to solve the PDE, we will apply the Fourier transform with respect to x, denoted by $\mathcal{F}_x$ and we treat t as parameter, i.e.

$$\hat{u}(\xi, t) = \mathcal{F}_x(u)(\xi, t) = \int_{\mathbb{R}} u(x,t) \mathrm{e}^{-2\pi i x \xi} \, dx.$$

Notice that if we assume smoothness and decay at infinity of the second derivatives, we have that

$$\mathcal{F}_x(\partial_{xx} u)(\xi, t) = (2\pi i)^2 \xi^2 \mathcal{F}(u)(\xi, t) = -(2\pi\xi)^2 \hat{u}(\xi, t)$$

Moreover, by dominated convergence we also have that

$$\mathcal{F}_x(\partial_{tt} u)(\xi, t) = \partial_{tt} \hat{u}(\xi, t).$$

Thus, when applying the Fourier transform to the equation, we formally get

$$\begin{cases} \hat{u}_{tt}(\xi, t) + (2\pi\xi)^2 \hat{u}(\xi, t) = 0 & (\xi, t) \in \mathbb{R} \times (0, \infty), \\ \hat{u}(\xi, 0) = \hat{g}(\xi), \hat{u}_t(\xi, t) = \hat{h}(\xi) & \xi \in \mathbb{R}. \end{cases}$$

Note that the PDE above only depends on derivatives in t. Thus, for each ξ the first PDE above is actually an ODE with respect to t and the solution is given by

$$\hat{u}(\xi, t) = \hat{g}(\xi)\cos(t2\pi|\xi|) + \frac{\hat{h}(\xi)}{2\pi|\xi|}\sin(t2\pi|\xi|).$$

Applying the Fourier inversion, we formally get

$$u(x,t) = \mathcal{F}_x^{-1}\left[\hat{g}(\xi)\cos(t2\pi|\xi|) + \frac{\hat{h}(\xi)}{2\pi|\xi|}\sin(t2\pi|\xi|)\right](x,t).$$

which one can check gives

$$u(x,t) = \frac{1}{2}\Big(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy\Big).$$

(ii)

- Assuming that $u \in C^2(\mathbb{R}^2 \times (0,\infty))$, we can differentiate E with respect to t : exchanging the order of integration and differentiation (see below), using the equation that u solves and integrating by parts (boundary terms vanishes because of the compact support)

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{1}{2}\int_{\mathbb{R}^n} 2u_t u_{tt}(x,t) + 2u_x(x,t)\partial_t u_x(x,t)\,dx \\ &= \int_{\mathbb{R}^n} u_t(x,t)u_{xx}(x,t) + u_x(x,t)\partial_t u_x(x,t)\,dx \\ &= \int_{\mathbb{R}^n} -\partial_x u_t(x,t)u_x(x,t) + u_x(x,t)\partial_t u_x(x,t)\,dx = 0.\end{aligned}$$

  Thus, $E(t) = E(0)$ for $t > 0$.

- Why can we exchange the order of differentiation and integration?

  We want to show that for all $t > 0$,

$$\frac{d}{dt}\int_{-\infty}^{\infty} F(x,t)dx = \int_{-\infty}^{\infty} \partial_t F(x,t)dx$$

  where $F(x,t) = (\partial_t u(x,t))^2 + (\partial_x u(x,t))^2$.

  Fix $t > 0$. Note that it is enough to show:

  (*) There is a compact set $K \subset \mathbb{R}$ such that the function $x \mapsto F(x,\tau)$ is supported in K for all $\tau \in (t/2, 3t/2)$.

  We can then apply the dominated convergence theorem.

  Now let's show (*). For this, we use that the general solution to the wave equation is

$$u(x,t) = \frac{1}{2}\Big(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy\Big)$$

  and we assume g, h are compactly supported in say $[-R, R]$. We have

$$\partial_t u(x,t) = \frac{1}{2}\Big(g'(x+t) - g'(x-t) + h(x+t) + h(x-t)\Big).$$

  Therefore, $\partial_t u(\cdot,\tau)$ is supported in $[-R-3t/2, R+3t/2]$ for all $\tau \in (t/2, 3t/2)$. Similarly, we can make the same claim for $\partial_x u$.

- For an alternative proof, one could use the representation formula derived formally in (i) and observe that under our assumptions, the representation formula holds pointwise. The conservation of the total energy can then be proven using this representation formula and Plancherel. Indeed,

$$\begin{aligned}\int_{\mathbb{R}} (u_x(x,t))^2 + (u_t(x,t))^2\,dx &= \int_{\mathbb{R}} (2\pi\xi)^2(\hat{u}(\xi,t))^2 + (\partial_t\hat{u}(\xi,t))^2\,d\xi \\ &= \int_{\mathbb{R}} (2\pi\xi)^2(\hat{g}(\xi))^2[\cos^2(t2\pi|\xi|) + \sin^2(t2\pi|\xi|)] \\ &+ \int_{\mathbb{R}} (\hat{h}(\xi))^2[\sin^2(t2\pi|\xi|) + \cos^2(t2\pi|\xi|)]\,d\xi \\ &= \int_{\mathbb{R}} \left[(2\pi\xi)^2(\hat{g}(\xi))^2 + (\hat{h}(\xi))^2\right]d\xi \\ &= \int_{\mathbb{R}} [(g_x(x))^2 + (h(x))^2]\,dx \\ &= E(0)\,.\end{aligned}$$

(iii) By (ii), it is enough to prove that

$$\lim_{t\to\infty}\int (u_x(x,t))^2\,dx = E(0)\,.$$

First of all, notice that by Plancharel and by the representation formula found in (i), we have

$$\begin{aligned}\int_{\mathbb{R}} (u_x(x,t))^2\,dx &= \int_{\mathbb{R}} (2\pi\xi)^2(\hat{u}(\xi))^2\,d\xi \\ &= \int_{\mathbb{R}} (2\pi\xi)^2(\hat{g}(\xi))^2\cos^2(t2\pi|\xi|) + (\hat{h}(\xi))^2\sin^2(t2\pi|\xi|)\,d\xi \\ &+ \int_{\mathbb{R}} 2\pi|\xi|\cos(t2\pi|\xi|)\sin(t2\pi|\xi|)(\hat{h}(\xi)\bar{\hat{g}}(\xi) + \hat{g}(\xi)\bar{\hat{h}}(\xi))\,d\xi\,.\end{aligned}$$

We claim that

$$\lim_{t\to\infty}\int_{\mathbb{R}} \cos(t2\pi|\xi|)\sin(t2\pi|\xi|)(2\pi|\xi|)(\hat{h}(\xi)\bar{\hat{g}}(\xi) + \hat{g}(\xi)\bar{\hat{h}}(\xi))\,d\xi = 0\,, \tag{11}$$

$$\lim_{t\to\infty}\int_{\mathbb{R}} (2\pi\xi)^2(\hat{g}(\xi))^2\cos^2(t2\pi|\xi|)\,dx = \frac{1}{2}\int_{\mathbb{R}} (2\pi\xi)^2(\hat{g}(\xi))^2\,d\xi\,, \tag{12}$$

$$\lim_{t\to\infty}\int_{\mathbb{R}} (\hat{h}(\xi))^2\sin^2(t2\pi|\xi|)\,d\xi = \frac{1}{2}\int_{\mathbb{R}} (\hat{h}(\xi))^2\,d\xi\,. \tag{13}$$

The claim of (iii) then follows from (11)–(13) and Plancherel. We first prove (11). We prove in fact that for every $f \in C_c^\infty(\mathbb{R})$, it holds

$$\lim_{t\to\infty}\int_{\mathbb{R}} \cos(t2\pi|\xi|)\sin(t2\pi|\xi|)f(\xi)\,d\xi = 0\,. \tag{14}$$

Observe however that

$$(2\pi|\xi|)(\hat{h}(\xi)\bar{\hat{g}}(\xi) + \hat{g}(\xi)\bar{\hat{h}}(\xi)) \in L^1(\mathbb{R})\,. \tag{15}$$

Indeed, note that due to Plancherel's identity, $(2\pi\xi)\hat{g} \in L^2(\mathbb{R})$ (since $\partial_x g \in C_c(\mathbb{R})$ and hence $\partial_x g \in L^2(\mathbb{R})$) and $\hat{h} \in L^2(\mathbb{R})$ (since $h \in C_c(\mathbb{R})$ and hence $h \in L^2(\mathbb{R})$), and thus we can easily deduce (15) from Hölder's inequality. We now deduce (11) from (14) by approximating the L^1 function $(2\pi\xi)(\hat{h}(\xi)\bar{\hat{g}}(\xi) + \hat{g}(\xi)\bar{\hat{h}}(\xi))$ by C_c^∞-functions (recall that C_c^∞ is dense in L^1!). Indeed, assuming (14) and using the density, we have that for any $\varepsilon > 0$, there is $f \in C_0^\infty(\mathbb{R})$ such that $\|(2\pi|\xi|)(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}) - f\|_{L^1(\mathbb{R})} < \varepsilon$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \cos(t2\pi|\xi|)\sin(t|2\pi\xi|)2\pi|\xi|(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}})\,d\xi\right| \\
&\leq \int_{\mathbb{R}} |\cos(t|2\pi\xi|)\sin(t|2\pi\xi|)|2\pi\xi|(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}) - f|\,d\xi + \int_{\mathbb{R}} \cos(t|2\pi\xi|)\sin(t|2\pi\xi|)f\,d\xi \\
&\leq \|(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}}) - f\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} \cos(t|2\pi\xi|)\sin(t|2\pi\xi|)f\,d\xi
\end{aligned}$$

Since the last term goes to 0 as $t \to \infty$, we deduce

$$\limsup_{t\to\infty} \left|\int_{\mathbb{R}^n} \cos(t|2\pi\xi|)\sin(t|2\pi\xi|)|2\pi\xi|(\hat{h}\bar{\hat{g}} + \hat{g}\bar{\hat{h}})\,d\xi\right| \leq \varepsilon$$

and since $\varepsilon > 0$ was arbitrary, we conclude (11) assuming (14). Finally, we prove (14). For $f \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}} \cos(t2\pi|\xi|)\sin(t2\pi|\xi|)f(\xi)\,d\xi &= \frac{1}{2}\int_{\mathbb{R}} \sin(2t(2\pi|\xi|))f(\xi)\,d\xi \\
&= \frac{1}{2}\int_0^\infty \sin(4\pi t\xi)(f(\xi) + f(-\xi))\,d\xi \\
&= -\frac{1}{8\pi t}\int_0^\infty \frac{d}{d\xi}(\cos(4t\pi\xi))(f(\xi) + f(-\xi))\,d\xi \\
&= \frac{1}{8\pi t}\int_0^\infty (\cos(4t\pi\xi))(f'(\xi) - f'(-\xi))d\xi.
\end{aligned}$$

where the last equality follows from integration by parts. Since $f \in C_c^\infty$, there is a constant C such that

$$\left|\int_{\mathbb{R}} \cos(t|2\pi\xi|)\sin(t|2\pi\xi|)f(\xi)\,d\xi\right| \leq \frac{C}{t}.$$

Thus, we deduce (14).

As for the claim (12), it follows from trigonometric identities that

$$\cos^2(t2\pi|\xi|) = \frac{1}{2}\left[\cos(2t(2\pi|\xi|)) + 1\right]$$

and therefore

$$\int_{\mathbb{R}} (2\pi\xi)^2(\hat{g}(\xi))^2\cos^2(t2\pi|\xi|)\,d\xi = \frac{1}{2}\int_{\mathbb{R}} (2\pi\xi)^2(\hat{g}(\xi))^2\left[\cos(2t(2\pi|\xi|)) + 1\right]\,d\xi\,.$$

Proceeding exactly as before, we show that the t-dependent contribution in the integral vanishes as $t \to \infty$ and we find

$$\lim_{t\to\infty} \int_{\mathbb{R}} (2\pi\xi)^2 (\hat{g}(\xi))^2 \cos^2(t2\pi|\xi|) = \frac{1}{2} \int_{\mathbb{R}} (2\pi\xi)^2 (\hat{g}(\xi))^2 \, d\xi$$

as claimed. We proceed analogously to prove (13).