Serie 12
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\theta \in [0, 2\pi)$. For $r \in (0,1)$ and $N \geq 1$ we define

$$S_N(r) = \sum_{n=1}^{N} r^n \cos(n\theta)$$

(i) Show that the sequence $\{S_N\}_{N\in\mathbb{N}}$ converges pointwise to a function S in $(0,1)$. Compute S explicitly.

(ii) For $r \in (0,1)$ and $n \geq 0$, deduce, by means of Fourier series, the value of

$$I_n(r) = \int_0^{2\pi} \frac{\cos(n\theta)}{1 - 2r\cos(\theta) + r^2} d\theta$$

Exercise 2.

(i) Find formally, using Fourier series, the solution $u = u(x,t)$ to the initial value problem

$$\begin{cases} u_t - u_{xx} = u & (x,t) \in (0,\pi) \times (0,\infty), \\ u(x,0) = f(x) & x \in (0,\pi), \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0. \end{cases}$$

(ii) Show that if $f \in L^1(0,\pi)$, then the function u obtained in (i) belongs to $C^0(]0,\pi[\times]0,\infty[)$. Actually, one can show that $u \in C^\infty(]0,\pi[\times]0,\infty[)$ but we don't prove it.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and even. Discuss under which assumptions on f the function u found in (i) satisfies

$$\lim_{t\to 0} u(x,t) = f(x) \text{ uniformly in } x\,.$$

In particular, observe, recalling (ii), that these assumptions guarantee that $u \in C^0(]0,\pi[\times[0,\infty[)$.

Exercise 3. We define the Schwartz space $\mathcal{S}(\mathbb{R})$ to be set of functions $f \in C^{\infty}(\mathbb{R})$ such that for all $k, l \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}} \left\{ \left|x^k\right| \left|f^{(l)}(x)\right| \right\} < \infty .$$

(i) Show that the function $f(x) = e^{-x^2}$ belongs to $\mathcal{S}(\mathbb{R})$.

(ii) Show that $C_c^{\infty}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq L^p(\mathbb{R})$ for $1 \le p \le \infty$.

(iii) Show that if $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

Exercise 4. The inequalities of Wirtinger and Poincaré establish a relationship between the L^2-norm of a function and the one of its derivative.

(i) If f is T-periodic, continuous and piecewise C^1 with $\int_0^T f(t)\, dt = 0$, show that

$$\int_0^T |f(t)|^2\, dt \le \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2\, dt,$$

with equality if and only if $f(t) = A\cos(2\pi t/T) + B\sin(2\pi t/T)$.

(ii) If f is as above and g is just C^1 and T-periodic, prove that

$$\left| \int_0^T \overline{f(t)} g(t)\, dt \right|^2 \le \frac{T^2}{4\pi^2} \left(\int_0^T |f(t)|^2\, dt \right) \left(\int_0^T |g'(t)|^2\, dt \right) .$$

(iii) For any compact interval $[a, b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2\, dt \le \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|^2\, dt.$$

Discuss the case of equality, and prove that the constant $(b-a)^2/\pi^2$ cannot be improved.

Hints:

- For (i), apply Parseval's identity.
- For (iii), extend f to be odd with respect to a and periodic of period $T = 2(b-a)$ so that its integral over an interval of length T is 0. Apply (i) to get the inequality and conclude that the equality holds if and only if

$$f(t) = A \sin\left(\pi\left(\frac{t-a}{b-a}\right)\right).$$

Exercise 5. Let $f, g \in L^1(\mathbb{R})$ and consider the Fourier transform of f given by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx = \lim_{N\to\infty} \int_{-N}^{N} f(x) e^{-2\pi i \xi x}\, dx.$$

Notice that the last equality holds due to the dominated convergence theorem. We already know from previous series that

- $\widehat{f}$ is well-defined,

- $\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$,
- $\lim_{|\xi|\to\infty} |\widehat{f}(\xi)| = 0$ and
- $\widehat{f}\colon \mathbb{R} \to \mathbb{C}$ is continuous.

Prove the following properties:

(i) **Linearity.** For any $a, b \in \mathbb{R}$ we have $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$.

(ii) **Translation.** If $a \in \mathbb{R}$ and $h(x) := f(x + a)$, then

$$\mathcal{F}(h)(\xi) = e^{2\pi i a\xi}\mathcal{F}(f)(\xi) \quad \forall \xi \in \mathbb{R}.$$

(iii) **Scaling.** If $a > 0$ and $h(x) := f(ax)$, then

$$\mathcal{F}(h)(\xi) = \frac{1}{a}\mathcal{F}(f)\left(\frac{\xi}{a}\right) \quad \forall \xi \in \mathbb{R}.$$

(iv) **Fourier transform of the derivative.** If, in addition, $f \in C^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, we have

$$\mathcal{F}(f')(\xi) = 2\pi i\xi\mathcal{F}(f)(\xi) \quad \forall \xi \in \mathbb{R}.$$

More generally, if $f \in C^n(\mathbb{R})$ and $f^{(k)} \in L^1(\mathbb{R})$ for all $k = 1, \ldots, n$, then

$$\mathcal{F}(f^{(n)})(\xi) = (2\pi i\xi)^n\mathcal{F}(f)(\xi) \quad \forall\, \xi \in \mathbb{R}.$$

Hint: Use without proving it the following fact : for any function $f \in \mathcal{C}^1 \cap L^1$ such that $f' \in L^1$, we have $\lim_{x\to\infty} |f(x)| = 0$.

(v) **Derivative of the Fourier transform.** If, in addition $h(x) := xf(x)$ belongs to $L^1(\mathbb{R})$, then the Fourier transform $\mathcal{F}(f)$ of f is differentiable and

$$\mathcal{F}(f)'(\xi) = -2\pi i\mathcal{F}(h)(\xi) \quad \forall\, \xi \in \mathbb{R}.$$

More generally, if $h_l(x) := x^l f(x)$ belongs to $L^1(\mathbb{R})$ for some l, then

$$\mathcal{F}(f)^{(l)}(\xi) = (-2\pi i)^l\mathcal{F}(h_l)(\xi) \quad \forall \xi \in \mathbb{R}.$$

(vi) **Product.** We have that

$$\int_{\mathbb{R}} \widehat{f}(x)g(x)\,dx = \int_{\mathbb{R}} f(x)\widehat{g}(x)\,dx\,.$$

Exercise 6. Let $f \in \mathcal{C}^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$ and $\forall x \in \mathbb{R},\ f'(x+1) = f(x)$. Prove that $f(x) = 0, \forall x \in \mathbb{R}$.

Exercise 7. The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f. You have already seen similar results for Fourier coefficients.

We define a *function of moderate decrease* as a function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}\ |f(x)| \leq \frac{A}{1+x^2}$ for some consant A.

(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function of moderate decrease whose Fourier transform $\hat{f}$ is continuous and satisfies :

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right), \text{ as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hlder condition of order α, i.e.

$$|f(x+h) - f(x)| \leq M|h|^{\alpha}, \text{ for some } M > 0 \text{ and } \forall x, h \in \mathbb{R}$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of 0. Prove that there is no $\alpha > 0$ such that $\hat{f}(\xi) = O(1/|\xi|^{1+\alpha})$ as $|\xi| \to \infty$.

Exercise 8. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, of moderate decrease, and such that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} e^{2xy}\, dx = 0$$

for all $y \in \mathbb{R}$, then $f = 0$.

Hint : Think of convolutions and gaussian kernels

Exercise 9 ($\star$)**.** Whereas it is easy to construct a continuous function which is not differentiable on a finite or even countable set, the question of whether or not there exists a nowhere differentiable, yet continuous function is much harder. The first example of such a function was given by Weierstrass in 1872.

Let's discuss here a slightly different example. The crucial feature of both examples is that the Fourier series skips many terms, we call such Fourier series lacunary. Let $0 < \alpha < 1$ and define

$$f_{\alpha}(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \mathrm{e}^{i 2^n x}.$$

It is clear that f_{α} is 1-periodic, continuous (since the series converges absolutely) and recall from Exercise 2 of Serie 11 that $f_{\alpha} \in C^{\alpha}$. Prove that f_{α} is differentiable nowhere.

Hint: Find an expression of $S_N f(x)$ in terms of the Cesaro means for $N = 2^{n-1}$ and find a lower bound on $|(S_{2N} f)'(x_0) - (S_N f)'(x_0)|$. Show that this lower bound is not compatible with f being differentiable in x_0.