Serie 12
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\theta \in [0, 2\pi)$. For $r \in (0,1)$ and $N \geq 1$ we define

$$S_N(r) = \sum_{n=1}^{N} r^n \cos(n\theta)$$

(i) Show that the sequence $\{S_N\}_{N\in\mathbb{N}}$ converges pointwise to a function S in $(0,1)$. Compute S explicitly.

(ii) For $r \in (0,1)$ and $n \geq 0$, deduce, by means of Fourier series, the value of

$$I_n(r) = \int_0^{2\pi} \frac{\cos(n\theta)}{1 - 2r\cos(\theta) + r^2} d\theta$$

Solution:

(i) Notice that $|r^n \cos(n\theta)| \leq r^n$ and that $\sum_{n\geq 1} r^n < +\infty$ when $r \in (0,1)$. Therefore, the sequence of real numbers $\{S_N(r)\}_N$ converges for every $r \in (0,1)$ (the series actually converges absolutely), which means that we can define the function S as follows:

$$S(r) = \sum_{n=1}^{\infty} r^n \cos(n\theta)$$

Recall that the real part operator $\Re : \mathbb{C} \to \mathbb{R}$ is linear and hence continuous. Therefore we have :

$$\begin{aligned} S(r) &= \sum_{n=1}^{\infty} \Re(r^n e^{in\theta}) = \Re\left(\sum_{n=1}^{\infty} (re^{i\theta})^n\right) \\ &= \Re\left(\frac{1}{1 - re^{i\theta}} - 1\right) = \Re\left(\frac{re^{i\theta}(1 - re^{-i\theta})}{1 - 2r\cos(\theta) + r^2}\right) \\ &= \frac{r\cos(\theta) - r^2}{1 - 2r\cos(\theta) + r^2} = -\frac{1}{2} + \frac{1}{2}\frac{1 - r^2}{1 - 2r\cos(\theta) + r^2} \end{aligned}$$

and thus

$$2S(r) = -1 + \frac{1-r^2}{1-2r\cos(\theta)+r^2}$$

(ii) For $r \in (0,1)$, let us consider the 2π-periodic function given by

$$f(\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

The previous question shows that this function is actually equal to :

$$f(\theta) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta).$$

which is actually the Fourier expansion in cosines of f (it makes sense since f is even). We can identify the coefficients: $a_0 = 2$ and $a_n = 2r^n$ for $n \geq 1$. and use the explicit formulae for this coefficients :

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad \forall n \geq 0$$

to obtain

$$\frac{1}{\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta)+r^2}d\theta = a_0 = 2$$

and

$$\frac{1}{\pi}\int_0^{2\pi} \frac{(1-r^2)\cos(n\theta)}{1-2r\cos(\theta)+r^2}d\theta = a_n = 2r^n$$

which yields

$$\boxed{I_n(r) = \frac{2\pi r^n}{1-r^2}, \quad \forall n \geq 0}$$

Exercise 2.

(i) Find formally, using Fourier series, the solution $u = u(x,t)$ to the initial value problem

$$\begin{cases} u_t - u_{xx} = u & (x,t) \in (0,\pi) \times (0,\infty), \\ u(x,0) = f(x) & x \in (0,\pi), \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0. \end{cases}$$

(ii) Show that if $f \in L^1(0,\pi)$, then the function u obtained in (i) belongs to $C^0(]0,\pi[\times]0,\infty[)$. Actually, one can show that $u \in C^\infty(]0,\pi[\times]0,\infty[)$ but we don't prove it.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and even. Discuss under which assumptions on f the function u found in (i) satisfies

$$\lim_{t\to 0} u(x,t) = f(x) \text{ uniformly in } x\,.$$

In particular, observe, recalling (ii), that these assumptions guarantee that $u \in C^0(]0,\pi[\times[0,\infty[)$.

Solution:

(i) *Step 1:*

The general way to find formally a solution of a PDE is to write $u(x,t) = \sum_{k\in\mathbb{Z}} c_k(t)e^{ikx}$ (notice that every function $u(x,t)$ can be written in this way, assuming for instance that $u(\cdot,t) \in L^2$ for any t) where $c_k : \mathbb{R} \to \mathbb{R}$ are functions depending on time to be determined and $f(x) = \sum_{k\in\mathbb{Z}} f_k e^{ikx}$. Plugging into the equation $u(x,t)$ and formally commuting the sum with the derivatives (∂_t and $\partial_x x$) we have

$$\sum_k \partial_t c_k(t)e^{ikx} + k^2 c_k(t)e^{ikx} - c_k(t)e^{ikx} = 0$$

and the last is true if and only if $c_k(t)$ solves the following ODEs

$$\begin{cases} \frac{d}{dt}c_k = -k^2 c_k + c_k \\ c_k(0) = f_k \end{cases}$$

for any $k \in \mathbb{Z}$. And one can solves explicitly the ODE obtaining that $c_k(t) = e^{(1-k^2)t}f_k$ for any $k \in \mathbb{Z}$. One can use the formula for a_k, b_k (the one for the basis with sinus and cosinus) from c_k or directly get the ODE for those coefficients from the PDE. If one would rigorously justify that the solution found is a solution to the PDE one would have to verify that $u(x,t) = \sum_k e^{(1-k^2)t} f_k$ is such that

- $\partial_t u \in C^0$, $\partial_{xx} u \in C^0$ and $u \in C^0$,
- we commute the derivative with the sum, namely

$$\partial_t \sum_k e^{(1-k^2)t} f_k = \sum_k \partial_t e^{(1-k^2)t} f_k$$

  and the same for ∂_x and ∂_{xx}
- $\lim_{t\to 0} u(t,x) = f(x)$ uniformly in x (which is the notion you gave in class to get the initial datum)
- $\partial_x u(0,t) = \partial_x u(\pi,t)$ for any $t > 0$ (which in general is not possible in the sense that the formal solution found does not have this property. For instance, in this example you can verify formally that $\partial_x u(0,t) = \partial_x u(\pi,t)$ for any $t > 0$ holds if $f \in L^1$ and it is even)

The other method is using the separation of variables. We ignore the initial condition: $u(x,0) = f(x), x \in]0,\pi[$ and focus on the two other equations. We use separation of variables and make the ansatz

$$u(x,t) = v(x)w(t).$$

The first equation then rewrites as $v(x)w'(t) - v''(x)w(t) = v(x)w(t)$ for all $(x,t) \in (0,\pi) \times (0,+\infty)$, which we can rewrite (provided that the denominator is non-zero) as

$$\frac{v''(x)}{v(x)} = \frac{w'(t) - w(t)}{w(t)} \quad \forall (x,t) \in (0,\pi) \times (0,+\infty)\,.$$

The latter can hold if and only if there exits $\lambda \in \mathbb{R}$ such that

$$\frac{v''(x)}{v(x)} \equiv -\lambda \equiv \frac{w'(t) - w(t)}{w(t)} \quad \forall (x,t) \in (0,\pi) \times (0,+\infty)\,.$$

This allows us to transform the first and third equations of our PDE in the two ODE's

$$\begin{cases} v''(x) + \lambda v(x) = 0 & \text{for all } x \in (0,\pi), \\ v'(0) = v'(\pi) = 0, \end{cases} \tag{1}$$

and

$$w'(t) = -(\lambda - 1) w(t) \quad \text{for all } t > 0\,. \tag{2}$$

Recall that the eigenvalues of (1) are $\lambda_n = n^2$ and the eigenfunctions are $v_n(x) = \cos(nx)$ for $n \geq 0$. When $\lambda = \lambda_n$, (2) is solved by $w_n(t) = \mathrm{e}^{-(n^2-1)t}$. Thus, for all $n \geq 0$

$$u_n(x,t) = v_n(x) w_n(t) = \cos(nx) \mathrm{e}^{-(n^2-1)t}$$

solves the first and third equation of our original PDE. Since any linear combination of solutions is also a solution as well, the general solution is given - formally - by

$$u(x,t) = \frac{a_0}{2} \mathrm{e}^t + \sum_{n=1}^{+\infty} a_n \cos(nx) \mathrm{e}^{-(n^2-1)t}, \tag{3}$$

where a_n are some constants. Observe that this solution is only a formal solution, since we did not discuss the convergence of the infinite series.

Step 2: We extend f to an even function on $(-\pi,\pi)$ by setting

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in (0,\pi), \\ f(-x) & \text{if } x \in (-\pi,0)\,. \end{cases} \tag{4}$$

Observe that $\tilde{f}(x) = f(x)$ for $x \in (0,\pi)$ and writing the Fourier series of $\tilde{f}$ in cosinus, we deduce that if we set

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx)\, dx, \quad n \geq 0\,,$$

then formally, for $t = 0$, u coincides with the Fourier series of $\tilde{f}$, which formally, coincides with $\tilde{f}$ pointwise and hence in particular with $f(x)$ for $x \in (0,\pi)$. Hence with this choice of the coefficients, (3) formally solves our PDE.

(ii) If $f \in L^1(0,\pi)$, then $\tilde{f} \in L^1(-\pi,\pi)$ and by Riemann-Lebesgue there is a constant α such that $|a_n| \leq \alpha$ for all $n \geq 0$. Thus,

$$\left| a_n \cos(nx) \mathrm{e}^{-(n^2-1)t} \right| \leq \alpha \mathrm{e}^{-(n^2-1)t} \quad n \geq 0\,.$$

Let $t_0 > 0$. Using the above bound, it is easy to see that when $(x,t) \in (0,\pi) \times (t_0,+\infty)$, the series (3) converges uniformly and absolutely. In particular, $u \in C^0(]0,\pi[\times]0,\infty[)$.

(iii) Since f is even, we have the Fourier series of f on $[-\pi, \pi]$ is given by

$$Ff(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

for $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$. Using the formal solution (3) found in (i), we have that

$$u(x,t) - f(x) = \frac{a_0}{2}\left(\mathrm{e}^t - 1\right) + \sum_{n=1}^{+\infty} a_n \cos(nx)\left(\mathrm{e}^{-(n^2-1)t} - 1\right). \tag{5}$$

Solution 1: $f \in C^4(\mathbb{R})$.
If $f \in C^4(\mathbb{R})$, we have that $a_n = O(n^{-4})$ and hence there exists a constant γ such that

$$|a_n| \le \frac{\gamma}{n^4}, \quad n \ge 1.$$

Recalling that $1 - e^{-x} \le x$ for $x > 0$, we estimate

$$0 \le \left|\mathrm{e}^{-(n^2-1)t} - 1\right| = 1 - \mathrm{e}^{-(n^2-1)t} \le (n^2-1)t, \quad n \ge 1.$$

We deduce

$$|u(x,t) - f(x)| \le t\left[\frac{|a_0|}{2} + \sum_{n=1}^{+\infty} |a_n|\left(n^2-1\right)\right] \le t\,\underbrace{\left[\frac{|a_0|}{2} + \gamma\sum_{n=1}^{+\infty}\frac{(n^2-1)}{n^4}\right]}_{<\infty},$$

which proves $\lim_{t\to 0} u(x,t) = f(x)$ uniformly in $x \in [0,\pi]$.

Solution 2: $f \in C^{1,\alpha}(\mathbb{R})$.
If $f \in C^{1,\alpha}(\mathbb{R})$ for some $0 < \alpha < 1$ (that is $f \in C^1(\mathbb{R})$ and $f' \in C^\alpha(\mathbb{R})$ - this is for instance guaranteed if $f \in C^2(\mathbb{R})$), then $a_n = O(n^{-(1+\alpha)})$ and hence there exists a constant γ such that

$$|a_n| \le \frac{\gamma}{n^{1+\alpha}}, \quad n \ge 1. \tag{6}$$

In particular, the series

$$\sum_{n=1}^{+\infty} a_n \cos(nx)\left(\mathrm{e}^{-(n^2-1)t} - 1\right) = -\sum_{n=1}^{+\infty} a_n \cos(nx)\left(1 - \mathrm{e}^{-(n^2-1)t}\right)$$

converges absolutely, uniformly in $t \in [0,1]$. Hence, we can exchange the limit and the summation and we deduce that

$$\lim_{t\to 0} -\sum_{n=1}^{+\infty} a_n \cos(nx)\left(1 - \mathrm{e}^{-(n^2-1)t}\right) = 0.$$

Hence, recalling (5), we have the pointwise convergence

$$\lim_{t\to 0} u(x,t) = f(x).$$

To show that the convergence is uniform in $x \in [0, \pi]$, we fix $\varepsilon > 0$. Using the triangular inequality, (5) and (6), we estimate, for a $N \geq 1$ yet to be chosen,

$$|u(x,t) - f(x)| \leq \frac{|a_0|}{2}(e^t - 1) + \gamma(1 - e^{-(N^2-1)t}) \sum_{n=1}^{N-1} \frac{1}{n^{1+\alpha}} + 2\gamma \sum_{n=N}^{\infty} \frac{1}{n^{1+\alpha}} .$$

We now choose $N = N(\varepsilon, \gamma) \geq 1$ large enough such that

$$2C \sum_{n=N}^{\infty} \frac{1}{n^{1+\alpha}} \leq \frac{\varepsilon}{2} .$$

Now, we choose $t_0 = t_0(N, \gamma) > 0$ small enough, such that for all $0 < t < t_0$

$$\frac{|a_0|}{2}(e^t - 1) + \gamma(1 - e^{-(N^2-1)t}) \sum_{n=1}^{N-1} \frac{1}{n^{1+\alpha}} \leq \frac{\varepsilon}{2} .$$

In particular, we have shown that for all $0 < t < t_0$, it holds $|u(x,t) - f(x)| \leq \varepsilon$ for all $x \in [0, \pi]$, hence the convergence is uniform on $[0, \pi]$.

Exercise 3. We define the Schwartz space $\mathcal{S}(\mathbb{R})$ to be set of functions $f \in C^\infty(\mathbb{R})$ such that for all $k, l \in \mathbb{N}$

$$\sup_{x \in \mathbb{R}} \left\{ \left| x^k \right| \left| f^{(l)}(x) \right| \right\} < \infty .$$

(i) Show that the function $f(x) = e^{-x^2}$ belongs to $\mathcal{S}(\mathbb{R})$.

(ii) Show that $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq L^p(\mathbb{R})$ for $1 \leq p \leq \infty$.

(iii) Show that if $f \in \mathcal{S}(\mathbb{R})$, then $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

Solution:

(i) One can show easily by induction that any derivative $f^{(l)}(x)$ takes the form

$$P_l(x)\mathrm{e}^{-x^2}$$

where P_l is a polynomial of degree l. Thus, in order to conclude we need to prove that

$$\sum_{x \in \mathbb{R}} \{|x^k f^{(l)}(x)|\} = \sup_{x \in \mathbb{R}} \{|x^k P_l(x)\mathrm{e}^{-x^2}|\} < \infty.$$

To do this, it suffices to show that the function $x \mapsto x^k P_l(x)\mathrm{e}^{-x^2}$ goes to 0 as $|x| \to \infty$. One can use de l'Hopital's rule to do this.

(ii) The first inclusion $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ is obvious because for any function $f \in C_c^\infty(\mathbb{R})$, $f^{(l)}$ is bounded by some C_l and the support of f is contained in some ball of radius R. Thus,

$$\sup_{x \in \mathbb{R}} \left\{ \left| x^k \right| \left| f^{(l)}(x) \right| \right\} \leq C_l R^k < \infty.$$

It is also trivial that $\mathcal{S}(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$. It only remains to prove $\mathcal{S}(\mathbb{R}) \subseteq L^p(\mathbb{R})$ for $p \in [1, \infty)$. If $f \in \mathcal{S}(\mathbb{R})$, we know that there is a constant C such that

$$\sup_{x \in \mathbb{R}} (1 + x^2)|f(x)| \leq C.$$

It follows that

$$\int_{\mathbb{R}} |f(x)|^p \, dx = \int_{\mathbb{R}} \frac{(1+x^2)|f(x)|^p}{(1+x^2)} \, dx \leq C^p \int_{\mathbb{R}} \frac{1}{(1+x^2)^p} \, dx \leq C^p \int_{\mathbb{R}} \frac{1}{(1+x^2)} \, dx < \infty.$$

(iii) For $k, l \in \mathbb{N}$, using the properties of the Fourier transform with respect to differentiation and multiplication (see for instance Exercise 5), we get

$$\xi^k \left(\frac{d}{d\xi} \right)^l \widehat{f}(\xi) = \widehat{g}(\xi),$$

with

$$g(x) = \frac{1}{(2\pi i)^k} \left(\frac{d}{dx} \right)^k [(-2\pi i x)^l f(x)].$$

Since $f \in \mathcal{S}(\mathbb{R})$, we also have $g \in \mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R})$. Finally,

$$\sup_{\xi \in \mathbb{R}} \left\{ \left| \xi^k \left(\frac{d}{d\xi} \right)^l \widehat{f}(\xi) \right| \right\} = \|\hat{g}\|_{L^\infty} \leq \|g\|_{L^1} < \infty,$$

and therefore $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

Exercise 4. The inequalities of Wirtinger and Poincaré establish a relationship between the L^2-norm of a function and the one of its derivative.

(i) If f is T-periodic, continuous and piecewise C^1 with $\int_0^T f(t)\, dt = 0$, show that

$$\int_0^T |f(t)|^2 \, dt \leq \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 \, dt,$$

with equality if and only if $f(t) = A\cos(2\pi t/T) + B\sin(2\pi t/T)$.

(ii) If f is as above and g is just C^1 and T-periodic, prove that

$$\left| \int_0^T \overline{f(t)} g(t) \, dt \right|^2 \leq \frac{T^2}{4\pi^2} \left(\int_0^T |f(t)|^2 \, dt \right) \left(\int_0^T |g'(t)|^2 \, dt \right).$$

(iii) For any compact interval $[a, b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2 \, dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|^2 \, dt.$$

Discuss the case of equality, and prove that the constant $(b-a)^2/\pi^2$ cannot be improved.

Hints:

- For (i), apply Parseval's identity.
- For (iii), extend f to be odd with respect to a and periodic of period $T = 2(b-a)$ so that its integral over an interval of length T is 0. Apply (i) to get the inequality and conclude that the equality holds if and only if

$$f(t) = A\sin\left(\pi\left(\frac{t-a}{b-a}\right)\right).$$

Solution:

(i) Observe that it is enough to prove the claim for $T = 1$ (otherwise, consider $\tilde{f}(x) := f(x/T)$). Since $\int_0^1 f(t)\,dt = 0$, $a_0 = 0$ and hence the Fourier series of f is of the form

$$Ff(x) := \sum_{k=1}^{\infty} \left[a_k \cos(2\pi kt) + b_k \sin(2\pi kt)\right].$$

Using Parseval's identity, we get

$$2\int_0^1 |f(t)|^2\,dt = \sum_{k=1}^{\infty}(a_k^2 + b_k^2). \tag{7}$$

The Fourier series of f' is given by

$$\sum_{k=1}^{\infty}\left[2\pi k b_k \cos(2\pi kt) + 2\pi k a_k \sin(2\pi kt)\right].$$

Note that the coeffcients of this Fourier series were not obtained by differentiating the Fourier series of f. Indeed, to find the Fourier series of f' as above we integrate the quantities

$$\int_0^1 f'(x)\cos(2\pi x)\,dx \quad \text{and} \quad \int_0^1 f'(x)\sin(2\pi x)\,dx$$

by parts. Again, by Parseval's identity,

$$2\int_0^1 |f'(t)|^2\,dt = \sum_{k=1}^{\infty}(2\pi k)^2(a_k^2 + b_k^2). \tag{8}$$

We conclude

$$\int_0^1 |f(t)|^2\,dt = \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{2}\sum_{k=1}^{\infty}k^2(a_k^2 + b_k^2) = \frac{1}{4\pi^2}\int_0^1 |f'(t)|^2\,dt. \tag{9}$$

It is obvious that the previous inequality is an equality if and only if $a_k = b_k = 0$ for all $k \ge 2$, i.e there is $A(= a_1)$ and $B(= b_1)$ such that $f(t) = A\cos(2\pi t) + B\sin(2\pi t)$.

(ii) Again, it suffices to prove the claim for $T = 1$ (otherwise consider $\tilde{f}(x) := f(x/T)$ and $\tilde{g}(x) := g(x/T)$ .Define the function $g_0\colon [0,1] \to \mathbb{R}$ by

$$g_0(t) = g(t) - \int_0^1 g(t)\,dt,$$

so that g_0 is continuous, C^1 and 1-periodic (since g is) and moreover

$$\int_0^1 g_0(t)\,dt = 0 \quad \text{and} \quad g_0'(t) = g'(t).$$

From (i), we have

$$\int_0^1 |g_0(t)|^2\,dt \leq \frac{1}{4\pi^2}\int_0^1 |g_0'(t)|^2\,dt = \frac{1}{4\pi^2}\int_0^1 |g'(t)|^2\,dt.$$

From Hölder's inequality, we have

$$\left|\int_0^1 \overline{f(t)}g_0(t)\,dt\right|^2 \leq \left(\int_0^1 |f(t)|^2\,dt\right)\left(\int_0^1 |g_0(t)|^2\,dt\right) \leq \frac{1}{4\pi^2}\left(\int_0^1 |f(t)|^2\,dt\right)\left(\int_0^1 |g'(t)|^2\,dt\right).$$

In addition, by construction of g_0 and by the hypothesis on f

$$\int_0^1 \overline{f(t)}g(t)\,dt = \int_0^1 \overline{f(t)}\left(g_0(t) + \int_0^1 g(s)\,ds\right) = \int_0^1 \overline{f(t)}g_0(t)\,dt + \underbrace{\int_0^1 \overline{f(t)}\,dt}_{=0}\int_0^1 g(s)\,ds$$

$$= \int_0^1 \overline{f(t)}g_0(t)\,dt.$$

We deduce

$$\left|\int_0^1 \overline{f(t)}g(t)\,dt\right|^2 \leq \frac{1}{4\pi^2}\left(\int_0^1 |f(t)|^2\,dt\right)\left(\int_0^1 |g'(t)|^2\,dt\right).$$

(iii) Define $h\colon [-(b-a),(b-a)] \to \mathbb{R}$ by

$$h(t) = \begin{cases} f(a+t) & \text{if } t \in [0,(b-a)], \\ -f(a-t) & \text{if } t \in [-(b-a),0]. \end{cases} \tag{10}$$

Observe that h is a continuous and odd function with respect to 0 because f is continuous with $f(a) = f(b) = 0$. We can extend it to a $2(b-a)$-periodic function on $\mathbb{R}$ (which we identify with h). Since h is an$2(b-a)$-periodic and odd function, we have

$$\int_0^{2(b-a)} h(t)\,dt = \int_{-(b-a)}^{(b-a)} h(t)\,dt = 0$$

so that from (i) we deduce

$$\int_0^{2(b-a)} |h(t)|^2\,dt \leq \frac{(b-a)^2}{\pi^2}\int_0^{2(b-a)} |h'(t)|^2\,dt\,. \tag{11}$$

It is easy to see that this equivalent to

$$\int_a^b |f(t)|^2\,dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.$$

Finally, we show that the constant cannot be improved. We know from (i) that the inequality

(11) is an equality if and only if h takes the form

$$h(t) = A\cos\left(\frac{\pi t}{(b-a)}\right) + B\sin\left(\frac{\pi t}{(b-a)}\right).$$

However, by construction, the function h is odd with respect to 0 which implies that $A = 0$. Therefore,

$$f(t) = B\sin\left(\pi\left(\frac{t-a}{b-a}\right)\right)$$

for some B. In particular, the constant cannot be improved.

Exercise 5. Let $f, g \in L^1(\mathbb{R})$ and consider the Fourier transform of f given by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx = \lim_{N\to\infty}\int_{-N}^{N} f(x)e^{-2\pi i\xi x}\,dx.$$

Notice that the last equality holds due to the dominated convergence theorem. We already know from previous series that

- $\widehat{f}$ is well-defined,
- $\|\widehat{f}\|_{L^\infty} \le \|f\|_{L^1}$,
- $\lim_{|\xi|\to\infty} |\widehat{f}(\xi)| = 0$ and
- $\widehat{f}\colon \mathbb{R}\to\mathbb{C}$ is continuous.

Prove the following properties:

(i) **Linearity.** For any $a, b \in \mathbb{R}$ we have $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$.

(ii) **Translation.** If $a \in \mathbb{R}$ and $h(x) := f(x+a)$, then

$$\mathcal{F}(h)(\xi) = e^{2\pi i a\xi}\mathcal{F}(f)(\xi) \quad \forall \xi \in \mathbb{R}.$$

(iii) **Scaling.** If $a > 0$ and $h(x) := f(ax)$, then

$$\mathcal{F}(h)(\xi) = \frac{1}{a}\mathcal{F}(f)\left(\frac{\xi}{a}\right) \quad \forall \xi \in \mathbb{R}.$$

(iv) **Fourier transform of the derivative.** If, in addition, $f \in C^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, we have

$$\mathcal{F}(f')(\xi) = 2\pi i\xi\mathcal{F}(f)(\xi) \quad \forall \xi \in \mathbb{R}.$$

More generally, if $f \in C^n(\mathbb{R})$ and $f^{(k)} \in L^1(\mathbb{R})$ for all $k = 1, \dots, n$, then

$$\mathcal{F}(f^{(n)})(\xi) = (2\pi i\xi)^n\mathcal{F}(f)(\xi) \quad \forall\, \xi \in \mathbb{R}.$$

Hint: Use without proving it the following fact : for any function $f \in \mathcal{C}^1 \cap L^1$ such that $f' \in L^1$, we have $\lim_{x\to\infty} |f(x)| = 0$.

(v) **Derivative of the Fourier transform.** If, in addition $h(x) := xf(x)$ belongs to $L^1(\mathbb{R})$, then the Fourier transform $\mathcal{F}(f)$ of f is differentiable and

$$\mathcal{F}(f)'(\xi) = -2\pi i \mathcal{F}(h)(\xi) \quad \forall\, \xi \in \mathbb{R}.$$

More generally, if $h_l(x) := x^l f(x)$ belongs to $L^1(\mathbb{R})$ for some l, then

$$\mathcal{F}(f)^{(l)}(\xi) = (-2\pi i)^l \mathcal{F}(h_l)(\xi) \quad \forall \xi \in \mathbb{R}.$$

(vi) **Product.** We have that

$$\int_{\mathbb{R}} \widehat{f}(x) g(x)\, dx = \int_{\mathbb{R}} f(x) \widehat{g}(x)\, dx\,.$$

Solution:

(i) The linearity follows from the linearity of the integral. Indeed,

$$\begin{aligned}\mathcal{F}(af+bg)(\xi) &= \int_{\mathbb{R}} (af(x)+bg(x))\mathrm{e}^{-2\pi i x\xi}\, dx = a\int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i x\xi}\, dx + b\int_{\mathbb{R}} g(x)\mathrm{e}^{-2\pi i x\xi}\, dx \\ &= a\mathcal{F}(f)(\xi) + b\mathcal{F}(g)(\xi)\,.\end{aligned}$$

(ii) First notice that for any $\varphi \in L^1(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \varphi(x)\, dx = \int_{\mathbb{R}} \varphi(x+a)\, dx \quad \forall a \in \mathbb{R}. \tag{12}$$

Indeed, the equality above is easy to prove for step functions. Subsequently, one can consider a general non-negative measurable function, approximate it by step functions and conclude with the monotone convergence theorem. For a general integrable function, we split into into positive and negative parts and we conclude (12) by applying the result for non-negative measurable functions to the positive and negative part. We obtain,

$$\mathcal{F}(h)(\xi) = \int_{\mathbb{R}} f(x+a)\mathrm{e}^{-2\pi i\xi x}\, dx \overset{(12)}{=} \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i\xi(x-a)}\, dx = \mathrm{e}^{2\pi i a\xi}\mathcal{F}(f)(\xi)\,.$$

(iii) Proceeding as in (ii), we can prove that for any $\varphi \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} \varphi(ax)\, dx = \frac{1}{a}\int_{\mathbb{R}} \varphi(x)\, dx \quad \forall a > 0. \tag{13}$$

We then obtain

$$\mathcal{F}(h)(\xi) = \int_{\mathbb{R}} f(ax)\mathrm{e}^{-2\pi i\xi x}\, dx \overset{(13)}{=} \frac{1}{a}\int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i\xi x/a}\, dx = \frac{1}{a}\mathcal{F}(f)(\xi/a)\,.$$

(iv) By the hint, $\lim_{x\to\infty} |f(x)| = 0$. In particular, for any $\varepsilon > 0$ there exists N_ε such that for all $x \geq N_\varepsilon$ we have $|f(x)| \leq \varepsilon$. Thus, integrating by parts, we obtain, for all $N \geq N_\varepsilon$

$$\int_{-N}^{N} f'(x)\mathrm{e}^{-2\pi i\xi x}\, dx = (f(x)\mathrm{e}^{-2\pi i\xi x})|_{-N}^{N} + \int_{-N}^{N} 2\pi i\xi f(x)\mathrm{e}^{-2\pi i\xi x}\, dx\,.$$

Therefore, we obtain

$$\left| \int_{-N}^{N} f'(x) e^{-2\pi i \xi x} \, dx - 2\pi i \xi \int_{-N}^{N} f(x) e^{-2\pi i \xi x} \, dx \right| \leq 2\varepsilon. \tag{14}$$

Since $f, f' \in L^1(\mathbb{R})$, we get, due to the dominated convergence theorem,

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i \xi x} \, dx = \lim_{N \to \infty} \int_{-N}^{N} f(x) e^{2\pi i \xi x} \, dx,$$
$$\mathcal{F}(f')(\xi) = \int_{\mathbb{R}} f'(x) e^{2\pi i \xi x} \, dx = \lim_{N \to \infty} \int_{-N}^{N} f'(x) e^{2\pi i \xi x} \, dx.$$

Therefore, letting $N \to \infty$ in (14), we have

$$\left| \int_{\mathbb{R}} f'(x) e^{-2\pi i \xi x} \, dx - 2\pi i \xi \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx \right| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude $\mathcal{F}(f')(\xi) = 2\pi i \xi \mathcal{F}(f)(\xi)$.

Now, we prove the general result by induction. Assume it holds for some n and prove it for $n+1$. Let $f \in C^{n+1}(\mathbb{R})$ and $f^{(k)} \in L^1(\mathbb{R})$ for all $k = 1, \ldots, n+1$. Then, since the result holds for n,

$$\mathcal{F}(f^{(n+1)})(\xi) = (2\pi i \xi)^n \mathcal{F}(f')(\xi) = (2\pi i \xi)^{n+1} \mathcal{F}(f)(\xi),$$

where the last equality follows from the case $n = 1$ which we already proved.

(v) We will prove that for every $\xi \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} \frac{\widehat{f}(\xi + \varepsilon) - \widehat{f}(\xi)}{\varepsilon} + 2\pi i \widehat{h}(\xi) = 0.$$

This proves both the differentiability of $\hat{f}$ and the claimed formula for its derivative. Let $\varepsilon > 0$ be arbitrary. We have

$$\frac{\widehat{f}(\xi + \varepsilon) - \widehat{f}(\xi)}{\varepsilon} + 2\pi i \widehat{h}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \left[\frac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon} + 2\pi i x \right] dx.$$

Notice that

$$\left| \frac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon} \right| \leq 2\pi |x|$$

and

$$\lim_{\varepsilon \to 0} \left[\frac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon} - 2\pi i x \right] = 0$$

pointwise. In order to be able to apply the dominated convergence theorem, notice that by assumption

$$\left| f(x) e^{-2\pi i \xi x} \left[\frac{e^{-2\pi i \varepsilon x} - 1}{\varepsilon} + 2\pi i x \right] \right| \leq 4\pi |x f(x)| = 4\pi |h(x)| \in L^1(\mathbb{R}).$$

By the dominated convergence theorem, we get,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i\xi x}\left[\frac{\mathrm{e}^{-2\pi i\varepsilon x}-1}{\varepsilon}+2\pi i x\right]dx=0,$$

which proves the result.

Now, in order to prove the formula for higher order derivatives, we use an induction argument. Assume by induction that the formula holds for some l and prove it for $l+1$. We assume that $h_{l+1}\in L^1(\mathbb{R})$. In order to apply the formula for l, we need to make sure that $h_l\in L^1(\mathbb{R})$. Indeed,

$$\begin{aligned}\int_R |x^l f(x)|\,dx &= \int_{|x|\le 1}|x|^l|f(x)|\,dx+\int_{|x|>1}|x|^l|f(x)|\,dx\\ &\le \int_{|x|\le 1}|f(x)|\,dx+\int_{|x|>1}|x|^{l+1}|f(x)|\,dx<\infty\end{aligned}$$

Thus, using the induction hypothesis and the case $l=1$ that we already proved, we have

$$\begin{aligned}\mathcal{F}(f)^{(l+1)}(\xi)&=\frac{d}{d\xi}[\mathcal{F}(f)^{(l)}(\xi)]=\frac{d}{d\xi}[(-2\pi i)^l\mathcal{F}(h_l)(\xi)]=(-2\pi i)^l\frac{d}{d\xi}[\mathcal{F}(h_l)(\xi)]\\ &=(-2\pi i)^{(l+1)}\mathcal{F}(h_{l+1})(\xi).\end{aligned}$$

(vi) First of all, notice that both integrals are finite and well-defined because $f,g\in L^1$ and $\widehat{f},\widehat{g}\in L^\infty$. More precisely, we have

$$\int_{\mathbb{R}}\widehat{f}(x)g(x)\,dx=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}f(y)\mathrm{e}^{-2\pi i xy}\,dy\right)g(x)\,dx$$

and

$$\int_{\mathbb{R}}f(y)\widehat{g}(y)\,dy=\int_{\mathbb{R}}f(y)\left(\int_{\mathbb{R}}g(x)\mathrm{e}^{-2\pi i xy}\,dx\right)dy.$$

In order to prove the result we will use Fubini's theorem. However, we first need to show that the function

$$(x,y)\mapsto f(y)g(x)\mathrm{e}^{-2\pi i xy}$$

is integrable on $\mathbb{R}^2$. Indeed, by Tonelli's theorem

$$\begin{aligned}\int_{\mathbb{R}^2}|f(y)g(x)\mathrm{e}^{-2\pi i xy}|\,d(x,y)&=\int_{\mathbb{R}^2}|f(y)||g(x)|\,d(x,y)=\left(\int_{\mathbb{R}}|f(y)|\,dy\right)\left(\int_{\mathbb{R}}|f(y)||g(x)|\,dx\right)\\ &=\|f\|_{L^1(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}<\infty.\end{aligned}$$

Thus, using Fubini's theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} \widehat{f}(x)g(x)\,dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y)\mathrm{e}^{-2\pi ixy}\,dy\right) g(x)\,dx \\ &= \int_{\mathbb{R}^2} f(y)\mathrm{e}^{-2\pi ixy} g(x)\,d(x,y) \\ &= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(x)\mathrm{e}^{-2\pi ixy}\,dx\right) dy \\ &= \int_{\mathbb{R}} f(y)\widehat{g}(y)\,dy.\end{aligned}$$

Exercise 6. Let $f \in \mathcal{C}^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$ and $\forall x \in \mathbb{R},\ f'(x+1) = f(x)$. Prove that $f(x) = 0, \forall x \in \mathbb{R}$.

Solution: Using the point (iv) of previous exercise, we know that $\hat{f'}(\xi) = 2\pi i\xi \hat{f}(\xi)$. Moreover, using the point (ii) of previous exercise, we also know that $\hat{f}(\xi) = e^{2\pi i\xi}\hat{f'}(\xi)$. Therefore you have

$$\begin{aligned}&\hat{f}(\xi) = 2\pi i\xi e^{2\pi i\xi}\hat{f}(\xi), \forall \xi \in \mathbb{R} \\ \Longrightarrow\ &\hat{f}(\xi) = 0, \text{ a.e.} \\ \Longrightarrow\ &f(x) = 0, \text{ a.e. because } \hat{f} \in L^1 \text{ so we can apply inverse Fourier transform} \\ \Longrightarrow\ &f(x) = 0,\ \forall x \in \mathbb{R}, \quad \text{because } f \text{ is continuous}\end{aligned}$$

Note : The second to last implication shows that the Fourier transform is actually an injective linear map from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

Exercise 7. The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f. You have already seen similar results for Fourier coefficients.

We define a *function of moderate decrease* as a function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}\ |f(x)| \leq \frac{A}{1+x^2}$ for some consant A.

(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function of moderate decrease whose Fourier transform $\hat{f}$ is continuous and satisfies :

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right), \text{ as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hlder condition of order α, i.e.

$$|f(x+h) - f(x)| \leq M|h|^{\alpha}, \text{ for some } M > 0 \text{ and } \forall x, h \in \mathbb{R}$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of 0. Prove that there is no $\alpha > 0$ such that $\hat{f}(\xi) = O(1/|\xi|^{1+\alpha})$ as $|\xi| \to \infty$.

Solution:

(i) Since $\alpha > 0$, $\hat{f} \in L^1$ and we can apply the Fourier inversion formula to get

$$f(x) = \int_{\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi,\ \forall x \implies f(x+h) - f(x) = \int_{\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}(e^{2\pi i \xi h} - 1)\, d\xi,\ \forall x$$

Let's further notice that $e^{2\pi i \xi h} - 1 = e^{\pi i \xi h}(e^{\pi i \xi h} - e^{-\pi i \xi h}) = 2ie^{\pi i \xi h}\sin(\pi \xi h) \implies |e^{2\pi i \xi h} - 1| = 2|\sin(\pi \xi h)|$ so :

$$\begin{aligned}
\Big|\frac{f(x+h) - f(x)}{h^\alpha}\Big| &\leq \frac{1}{|h|^\alpha}\int_{-\infty}^{\infty} \frac{A|e^{2\pi i \xi h} - 1|}{1 + |\xi|^{1+\alpha}}\, d\xi \\
&\leq \frac{4A}{|h|^\alpha}\int_0^{\infty} \frac{|\sin(\pi \xi h)|}{1 + |\xi|^{1+\alpha}}\, d\xi \\
&= \frac{4A}{\pi}\int_0^{\infty} \frac{|\sin(u)|}{|h|^{1+\alpha} + u^{1+\alpha}}\, du \\
&\leq \frac{4A}{\pi}\left(\int_0^1 \frac{\left|\frac{\sin(u)}{u}\right|}{u^\alpha}\, du + \int_1^\infty \frac{1}{u^{1+\alpha}}\, du\right) \\
&\leq \frac{4A}{\pi}\left(\int_0^1 \frac{1}{u^\alpha}\, du + \int_1^\infty \frac{1}{u^{1+\alpha}}\, du\right) \\
&< \infty
\end{aligned}$$

(ii) We have $\forall \alpha > 0$ fixed :

$$\frac{|f(h) - f(0)|}{|h|^\alpha} = \frac{1}{-|h|^\alpha \log|h|} \underset{h\to 0}{\longrightarrow} \infty$$

so f is not α-Hlder continuous for any $\alpha > 0$. Since f is of moderate decrease (it is continuous on $\mathbb{R}$ and vanishes at ∞), $\hat{f}$ cannot verify the stated condition, or otherwise it would contradict the first question.

Exercise 8. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, of moderate decrease, and such that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} e^{2xy}\, dx = 0$$

for all $y \in \mathbb{R}$, then $f = 0$.
Hint : Think of convolutions and gaussian kernels

Solution: Let $g(x) = e^{-x^2}$. Then $\forall x \in \mathbb{R}$:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2}\, dx = e^{-x^2}\int_{-\infty}^{\infty} f(x) e^{-y^2} e^{2xy}\, dx = 0$$

This implies that $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi) = \hat{f}(\xi)\sqrt{\pi}e^{-\pi^2\xi^2} = 0,\ \forall \xi \in \mathbb{R}$ which in turn implies that $\hat{f} = 0 \in L^1$. By Fourier inversion formula, we can conclude that $f = 0$.

Exercise 9 ($\star$)**.** Whereas it is easy to construct a continuous function which is not differentiable on a finite or even countable set, the question of whether or not there exists a nowhere differentiable, yet continuous function is much harder. The first example of such a function was given by Weierstrass in 1872.

Let's discuss here a slightly different example. The crucial feature of both examples is that the Fourier series skips many terms, we call such Fourier series lacunary. Let $0 < \alpha < 1$ and define

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \mathrm{e}^{i2^n x}.$$

It is clear that f_α is 1-periodic, continuous (since the series converges absolutely) and recall from Exercise 2 of Serie 11 that $f_\alpha \in C^\alpha$. Prove that f_α is differentiable nowhere.

Hint: Find an expression of $S_N f(x)$ in terms of the Cesaro means for $N = 2^{n-1}$ and find a lower bound on $|(S_{2N}f)'(x_0) - (S_N f)'(x_0)|$. Show that this lower bound is not compatible with f being differentiable in x_0.

Solution: By contradiction, assume that f is differentiable in x_0. For $N \in \mathbb{N}$ we denote by

$$\Phi_N f(x) = \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x)$$

the Nth Cesaro mean. We proceed in 3 steps.

Step 1: For $N = 2^{n-1}$, we can write $S_N f(x) = 2\Phi_{2N} f(x) - \Phi_N f(x)$.

Indeed, observe that $S_N f(x) = S_{2^{n-1}} f(x) = S_k f(x)\ \forall k \in [N, 2^n - 1]$ due to the lacunary Fourier series. Thus,

$$\begin{aligned} S_N f(x) &= \frac{1}{N} \sum_{k=N}^{2N-1} S_k f(x) \\ &= \frac{1}{N} \sum_{k=0}^{2N-1} S_k f(x) - \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x) \\ &= 2\frac{1}{2N} \sum_{k=0}^{2N-1} S_k f(x) - \frac{1}{N} \sum_{k=0}^{N-1} S_k f(x) \\ &= 2\Phi_{2N} f(x) - \Phi_N f(x). \end{aligned}$$

Step 2: For $N = 2^{n-1}$, $S_{2N} f(x) - S_N f(x) = 2^{-n\alpha} \mathrm{e}^{i2^n x}$ and hence

$$|(S_{2N}f)'(x_0) - (S_N f)'(x_0)| = 2^{n(1-\alpha} \geq cN^{1-\alpha} \quad \textit{for some } c = c(\alpha) > 0. \tag{15}$$

Observe due to the lacunary Fourier series that $S_{2N} f(x) - S_N f(x) = 2^{n\alpha} \mathrm{e}^{i2^n x}$ and the righthand side is differentiable in x_0 with derivative

$$(S_{2N}f)'(x_0) - (S_N f)'(x_0) = 2^{n(1-\alpha)} \mathrm{e}^{i2^n x} = N^{1-\alpha} 2^{(1-\alpha)} \mathrm{e}^{i2^n x}.$$

Thus, taking $c = 2^{1-\alpha}$ gives the result.

Step 3: Using Step 1 and the differentiability of f in x_0, we find a contradiction with Step 2. From the equality we established in step 1, we have

$$S_{2N}f(x) - S_N f(x) = 2\Phi_{4N} f(x) - 3\Phi_{2N} f(x) + \Phi_N f(x). \tag{16}$$

Recall from Exercise 1 of Serie 9 that

$$(\Phi_N f)(x_0) = \int_{-\pi}^{\pi} F_N(x_0 - t) f(t)\, dt\,,$$

where F_N is the 2π-periodic version of the Fejer kernel. In particular, this means that

$$(\Phi_N f)'(x_0) = \int_{-\pi}^{\pi} F_N'(x_0 - t) f(t)\, dt = \int_{-\pi}^{\pi} F_N'(t) f(x_0 - t)\, dt.$$

Since F_N is 2π-periodic

$$\int_{-\pi}^{\pi} F_N'(t)\, dt = 0$$

and therefore

$$(\Phi_N f)'(x_0) = \int_{-\pi}^{\pi} F_N'(t)(f(x_0 - t) - f(x_0))\, dt.$$

Now, we prove that there is $C > 0$ such that

$$|(\Phi_N f)'(x_0)| \le C \int_{-\pi}^{\pi} |F_N'(t)||t|\, dt. \tag{17}$$

Since we assume f to be differentiable in x_0, there is $\varepsilon, \delta > 0$ such that $|f(x_0 - t) - f(x_0)| \le \varepsilon |t|$ for all t such that $|t| \le \delta$. For any t such that $|t| > \delta$,

$$|f(x_0 - t) - f(x_0)| \le 2\|f\|_{L^\infty} \le \frac{2\|f\|_{L^\infty}}{\delta}|t|.$$

Taking $C = \max\{\varepsilon, 2\|f\|_{L^\infty}/\delta\}$, gives (17). Now we prove that there is A such that

$$|F_N'(t)| \le AN^2 \quad \text{and} \quad |F_N'(t)| \le \frac{A}{|t|^2} \quad \forall t \in [-\pi, \pi].$$

For the first inequality, recall that F_N is a trigonometric polynomial of degree N whose coefficients are less than 1. Thus the derivative F_N' is trigonometric polynomial whose coefficients are less than N. Thus, $|F_N'(t)| \le (2N+1)N \le 3N^2$. For the second inequality, let us compute the derivative of the Fejer kernel. We have

$$F_N'(t) = \frac{\sin(Nt/2)\cos(Nt/2)}{\sin^2(t/2)} - \frac{1}{N}\frac{\cos(t/2)\sin^2(Nt/2)}{\sin^3(t/2)}.$$

Note that $|\sin(x)| \le |x|$ for all x and $|\sin(x)| \ge |x|/2$ for all $x \in [-\pi/2, \pi/2]$. Therefore,

$$F_N'(t) \le \frac{1}{\frac{1}{4}|t/2|} + \frac{1}{N}\frac{|Nt/2|}{\frac{1}{8}|t/2|^3} = \frac{16}{|t|^2} + \frac{8|t/2|}{|t/2|^3} = \frac{32}{|t|^2}$$

so that taking $A = 32$ we have the desired result. Using these inequalities, we get

$$\begin{aligned}
|(\Phi_N f)'(x_0)| &\leq C \int_{-\pi}^{\pi} |F_N'(t)||t| \, dt \\
&\leq C \int_{|t|\geq 1/N} |F_N'(t)||t| \, dt + C \int_{|t|\leq 1/N} |F_N'(t)||t| \, dt \\
&\leq CA \int_{\pi \geq |t| \geq 1/N} \frac{1}{|t|} \, dt + CAN \int_{|t|\leq 1/N} 1 \, dt \\
&= 2CA(\log(N) + \log(\pi)) + CAN \frac{2}{N} \\
&\leq C_1 \log(N) \quad \text{ for some } C_1 \text{ when } N \text{ is large enough.}
\end{aligned}$$

Finally, differentiating (16) and evaluating in $x = x_0$ gives

$$\begin{aligned}
|(S_{2N} f)'(x_0) - (S_N f)'(x_0)| &= |2(\Phi_{4N} f)'(x_0) - 3(\Phi_{2N} f)'(x_0) + (\Phi_N)' f(x_0)| \\
&\leq 2C_1 \log(4N) + 3C_1 \log(2N) + C_1 \log(N) \\
&\leq C_2 \log(N) \quad \text{ for some } C_2 \text{ when } N \text{ is large enough.}
\end{aligned}$$

This contradicts (15) and proves that f is not differentiable in x_0 and since x_0 was arbitrary, f is differentiable nowhere.