Serie 11

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $-\infty < a < b < \infty$ and $\alpha \in (0,1]$. For $f \in C^{0,\alpha}([a,b])$, we define $\|f\|_{C^{0,\alpha}([a,b])} := \|f\|_{C^0([a,b])} + [f]_{C^{0,\alpha}([a,b])}$, where

$$[f]_{C^{0,\alpha}([a,b])} := \sup_{a \le x \neq y \le b} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

(i) Show that $\|\cdot\|_{C^{0,\alpha}([a,b])}$ is a norm on $C^{0,\alpha}([a,b])$.

(ii) Let $f, g \in C^{0,\alpha}([a,b])$. Show that their product $fg \in C^{0,\alpha}([a,b])$.

(iii) Show that if $0 < \alpha \le \beta \le 1$, then

$$C^1([a,b]) \subset C^{0,1}([a,b]) \subseteq C^{0,\beta}([a,b]) \subseteq C^{0,\alpha}([a,b]) \subset C^0([a,b]).$$

(iv) Let $\alpha \in (0,1]$ and define $f_\alpha \colon [0,1] \to \mathbb{R}$ by $f_\alpha(x) := x^\alpha$. Show that $f_\alpha \in C^{0,\alpha}([0,1])$.

(v) Show that the function $f \colon [0,1/2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ -1/\log x & \text{if } x \in (0,1/2], \end{cases}$$

is continuous but not Hölder continuous for any $\alpha \in (0,1]$.

Hint: For (iv), compute the quantity $\sup_{t\in(0,1)} \left\{ \frac{1-t^\alpha}{(1-t)^\alpha} \right\}$ and use it to deduce the result.

Exercise 2. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is 1 periodic and of class C^k. We use the notation $\hat{f}_n = c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx$. Show that

$$\widehat{f}(n) = o(1/|n|^k),$$

that is $|n|^k \widehat{f}(n)$ goes to 0 as $|n| \to \infty$.

Exercise 3. Let f be 2π-periodic and integrable on $[-\pi, \pi]$. We use the notation $\hat{f}_n = c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}dx$.

(i) Show that

$$\widehat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)\mathrm{e}^{-inx}\,dx$$

and hence

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]\mathrm{e}^{-inx}\,dx.$$

(ii) Now assume that f satisfies a Hölder condition of order $0 < \alpha < 1$, namely that there exists $C > 0$ such that

$$|f(x+h) - f(x)| \leq C|h|^{\alpha} \quad \forall x, h \in \mathbb{R}\,.$$

Use (i) to show that

$$\widehat{f}(n) = O(1/|n|^{\alpha}).$$

(iii) Prove that the result cannot be improved by showing that for $0 < \alpha < 1$ fixed, the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}\mathrm{e}^{i2^k x}, \tag{3}$$

satisfies

$$|f(x+h) - f(x)| \leq C|h|^{\alpha} \quad \forall h \in \mathbb{R}$$

and $\widehat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$.

Hint: For (iii), break up the sum as follows

$$f(x+h) - f(x) = \sum_{2^k \leq 1/|h|} + \sum_{2^k > 1/|h|}\,.$$

To estimate the first sum use the fact that $|1 - \mathrm{e}^{i\theta}| \leq |\theta|$ whenever θ is small. To estimate the second sum, use the obvious inequality $|\mathrm{e}^{ix} - \mathrm{e}^{iy}| \leq 2$.

Exercise 4.

(i) Compute the Fourier series of the 2π-periodic odd function f defined by $f(x) = x(\pi - x)$ on $[0, \pi]$.

(ii) Using (i) and Parseval's identity, deduce the value of the series

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}.$$

Exercise 5. Let $f \in C^0([0,1])$ be a 1-periodic function and τ an irrational number. Show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\tau) = \int_0^1 f(x)\,dx$$

and that this result does not hold when τ is rational.
Hint: Begin by showing the result for functions of the form $f(x) = \mathrm{e}^{2\pi i k x}$ for some $k \in \mathbb{Z}$ and conclude by approximation.

Exercise 6 ($\star$)**.** Construct a continuous function whose Fourier series diverges at some point.
Hint: You may work on $[-\pi, \pi]$. For $N \in \mathbb{N}$, we define $\psi_N(x) := \operatorname{sgn}(x)\sin((N+\frac{1}{2})x)$ and choose the ansatz $f(x) := \sum_{k=1}^{\infty} a_k \psi_{N_k}(x)$ for a sequence $\{a_k\}_{k=1}^{\infty}$ of positive real numbers (what assumption on $\{a_k\}$ do you need?) and sequence $\{N_k\}_{k=1}^{\infty}$ of integers. Choose the sequence N_k carefully, depending on $\{a_k\}_k$, such that the Fourier series diverges in some point.