Serie 11

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $-\infty < a < b < \infty$ and $\alpha \in (0,1]$. For $f \in C^{0,\alpha}([a,b])$, we define $\|f\|_{C^{0,\alpha}([a,b])} := \|f\|_{C^0([a,b])} + [f]_{C^{0,\alpha}([a,b])}$, where

$$[f]_{C^{0,\alpha}([a,b])} := \sup_{a \le x \ne y \le b} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

(i) Show that $\|\cdot\|_{C^{0,\alpha}([a,b])}$ is a norm on $C^{0,\alpha}([a,b])$.

(ii) Let $f, g \in C^{0,\alpha}([a,b])$. Show that their product $fg \in C^{0,\alpha}([a,b])$.

(iii) Show that if $0 < \alpha \le \beta \le 1$, then

$$C^1([a,b]) \subset C^{0,1}([a,b]) \subseteq C^{0,\beta}([a,b]) \subseteq C^{0,\alpha}([a,b]) \subset C^0([a,b]).$$

(iv) Let $\alpha \in (0,1]$ and define $f_\alpha \colon [0,1] \to \mathbb{R}$ by $f_\alpha(x) := x^\alpha$. Show that $f_\alpha \in C^{0,\alpha}([0,1])$.

(v) Show that the function $f \colon [0,1/2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ -1/\log x & \text{if } x \in (0,1/2], \end{cases}$$

is continuous but not Hölder continuous for any $\alpha \in (0,1]$.

Hint: For (iv), compute the quantity $\sup_{t\in(0,1)} \left\{ \dfrac{1-t^\alpha}{(1-t)^\alpha} \right\}$ and use it to deduce the result.

Solution:

(i) Let $f, g \in C^{0,\alpha}([a,b])$ and $a \in \mathbb{R}$. Since $\|\cdot\|_{C^0([a,b])}$ is a norm on $C^0([a,b])$, we have that $\|f\|_{C^{0,\alpha}([a,b])} = 0$ implies $f = 0$. On the contrary, if $f = 0$ then $\|f\|_{C^{0,\alpha}([a,b])} = 0$. Thus,

$$\|f\|_{C^{0,\alpha}([a,b])} = 0 \quad \text{if and only if} \quad f = 0.$$

Then, since both $\|\cdot\|_{C^0([a,b])}$ and $[f]_{C^{0,\alpha}([a,b])}$ are absolutely homogeneous, we have

$$\|af\|_{C^{0,\alpha}([a,b])} = |a|\|f\|_{C^{0,\alpha}([a,b])}.$$

It remains to prove that

$$\|f+g\|_{C^{0,\alpha}([a,b])} \leq \|f\|_{C^{0,\alpha}([a,b])} + \|g\|_{C^{0,\alpha}([a,b])}. \tag{1}$$

We already have

$$\|f+g\|_{C^0([a,b])} \leq \|f\|_{C^0([a,b])} + \|g\|_{C^0([a,b])},$$

since $\|\cdot\|_{C^0([a,b])}$ is a norm. Moreover, for any $a \leq x \neq y \leq b$, we have

$$\begin{aligned}\frac{|(f+g)(x)-(f+g)(y)|}{|x-y|^\alpha} &= \frac{|f(x)+g(x)-f(y)-g(y)|}{|x-y|^\alpha}\\ &\leq \frac{|f(x)-f(y)|}{|x-y|^\alpha} + \frac{|g(x)-g(y)|}{|x-y|^\alpha}\\ &\leq [f]_{C^{0,\alpha}([a,b])} + [g]_{C^{0,\alpha}([a,b])}.\end{aligned}$$

Therefore, by taking the supremum over all x and y, we obtain

$$[f+g]_{C^{0,\alpha}([a,b])} \leq [f]_{C^{0,\alpha}([a,b])} + [g]_{C^{0,\alpha}([a,b])},$$

which gives (1).

(ii) It is clear that $fg \in C^0([a,b])$ and $\|fg\|_{C^0([a,b])} \leq \|f\|_{C^0([a,b])}\|g\|_{C^0([a,b])}$. In addition, for any $a \leq x \neq y \leq b$, we have

$$\begin{aligned}\frac{|f(x)g(x)-f(y)g(y)|}{|x-y|^\alpha} &= \frac{|f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)|}{|x-y|^\alpha}\\ &\leq \|f\|_{C^0([a,b])}\frac{|g(x)-g(y)|}{|x-y|^\alpha} + \|g\|_{C^0([a,b])}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\\ &\leq \|f\|_{C^0([a,b])}[g]_{C^{0,\alpha}([a,b])} + \|g\|_{C^0([a,b])}[f]_{C^{0,\alpha}([a,b])}.\end{aligned}$$

By taking the supremum over all $x \neq y$, we obtain

$$[fg]_{C^{0,\alpha}([a,b])} \leq \|f\|_{C^0([a,b])}[g]_{C^{0,\alpha}([a,b])} + \|g\|_{C^0([a,b])}[f]_{C^{0,\alpha}([a,b])}.$$

We therefore get that $fg \in C^{0,\alpha}([a,b])$ and

$$\begin{aligned}\|fg\|_{C^{0,\alpha}([a,b])} &= \|fg\|_{C^0([a,b])} + [fg]_{C^{0,\alpha}([a,b])}\\ &\leq \|f\|_{C^0([a,b])}\|g\|_{C^0([a,b])} + \|f\|_{C^0([a,b])}[g]_{C^{0,\alpha}([a,b])} + \|g\|_{C^0([a,b])}[f]_{C^{0,\alpha}([a,b])}\\ &\leq \|f\|_{C^{0,\alpha}([a,b])}\|g\|_{C^{0,\alpha}([a,b])}.\end{aligned}$$

(iii) (a) *We prove* $C^{0,\alpha}([a,b]) \subset C^0([a,b])$*:* Assume that $f \in C^{0,\alpha}([a,b])$. We then have

$$|f(x) - f(y)| \leq [f]_{C^{0,\alpha}([a,b])}|x-y|^\alpha \quad \forall x,y \in [a,b],$$

which implies that f is continuous. Thus, $f \in C^0([a,b])$.

(b) *We prove* $C^{0,\beta}([a,b]) \subset C^{0,\alpha}([a,b])$*:* For any $f \in C^{0,\beta}([a,b])$, we have

$$\begin{aligned}\sup_{a\leq x\neq y\leq b}\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right\} &= \sup_{a\leq x\neq y\leq b}\left\{\frac{|f(x)-f(y)||x-y|^{\beta-\alpha}}{|x-y|^\beta}\right\} \\ &\leq [f]_{C^{0,\beta}([a,b])}(b-a)^{\beta-\alpha}.\end{aligned}$$

From this it follows that $f \in C^{0,\alpha}([a,b])$ and

$$\begin{aligned}\|f\|_{C^{0,\alpha}([a,b])} &= \|f\|_{C^0([a,b])} + [f]_{C^{0,\alpha}([a,b])} \\ &\leq \|f\|_{C^0([a,b])} + [f]_{C^{0,\beta}([a,b])}(b-a)^{\beta-\alpha} \\ &\leq (1+(b-a)^{\beta-\alpha})\|f\|_{C^{0,\beta}([a,b])}.\end{aligned}$$

(c) *We prove* $C^{0,1}([a,b]) \subset C^{0,\beta}([a,b])$*:* This is a just special case of (b).

(d) *We prove* $C^1([a,b]) \subset C^{0,1}([a,b])$*:* Let $f \in C^1([a,b])$. For any $x,y \in [a,b]$, we have

$$\begin{aligned}f(x) - f(y) &= \int_0^1 \frac{d}{dt}\left[f(y+t(x-y))\right]\,dt \\ &= \int_0^1 f'(y+t(x-y))(x-y)\,dt.\end{aligned}$$

Hence we have

$$|f(x)-f(y)| \leq \max_{z\in[x,y]}|f'(z)||x-y| \leq \|f\|_{C^1([a,b])}|x-y|,$$

from which we deduce that $f \in C^{0,1}([a,b])$ and $[f]_{C^{0,1}([a,b])} \leq \|f\|_{C^1([a,b])}$. As a consequence,

$$\|f\|_{C^{0,1}([a,b])} = \|f\|_{C^0([a,b])} + [f]_{C^{0,1}([a,b])} \leq 2\|f\|_{C^1([a,b])}.$$

(iv) We begin by computing the quantity $\sup_{t\in(0,1)}\left\{\frac{1-t^\alpha}{(1-t)^\alpha}\right\}$. We claim that for all $\alpha \in (0,1]$, we have

$$g(t) = 1 - t^\alpha - (1-t)^\alpha \leq 0 \quad \forall t \in [0,1]. \tag{2}$$

Indeed, notice that g is convex (this can be shown by computing the second derivative and noting that it is positive) and $g(0) = g(1) = 0$ which implies (2), so that

$$\sup_{t\in(0,1)}\left\{\frac{1-t^\alpha}{(1-t)^\alpha}\right\} \leq \sup_{t\in(0,1)}\left\{\frac{(1-t)^\alpha}{(1-t)^\alpha}\right\} = 1.$$

From this we obtain

$$\sup_{0\le x\neq y\le 1}\left\{\frac{|f_\alpha(x)-f_\alpha(y)|}{|x-y|^\alpha}\right\} = \sup_{1\ge x>y\ge 0}\left\{\frac{|x^\alpha-y^\alpha|}{|x-y|^\alpha}\right\}$$
$$= \sup_{1\ge x>y\ge 0}\left\{\frac{|x^\alpha||(1-(y/x)^\alpha)|}{|x^\alpha||1-y/x|^\alpha}\right\}$$
$$= \sup_{t\in(0,1)}\left\{\frac{1-t^\alpha}{(1-t)^\alpha}\right\} = 1.$$

Thus, $f_\alpha \in C^{0,\alpha}([a,b])$.

(v) We easily see that f is continuous since $\log x \to -\infty$ as $x \to 0$. Now assume for a contradiction that there is $\alpha \in (0,1]$ such that $f \in C^{0,\alpha}([0,\frac{1}{2}])$. In other words, there exists $C_\alpha > 0$ such that

$$\frac{-1}{\log x} = |f(x)-f(0)| \le C_\alpha x^\alpha \qquad \forall x \in (0,1/2)\,.$$

But this implies that

$$\frac{-1}{C_\alpha} \ge x^\alpha \log x \qquad \forall x \in (0,1/2)\,,$$

which is impossible since $\lim_{x\to 0} x^\alpha \log x = 0$.

Exercise 2. Suppose that $f:\mathbb{R}\to\mathbb{C}$ is periodic and of class C^k. Show that

$$\widehat{f}(n) = o(1/|n|^k),$$

that is $|n|^k\widehat{f}(n)$ goes to 0 as $|n|\to\infty$.

Solution: Let $f \in C^k$ be L-periodic. We may assume without loss of generality that $L=1$ (if not, consider $\tilde{f}(x) := f(Lx)$). Using integration by parts, we get

$$\widehat{f}(n) = \int_0^1 f(x)\mathrm{e}^{-2\pi inx}\,dx = \left[f(x)\frac{-1}{2\pi in}\mathrm{e}^{-2\pi inx}\Big|_0^1 - \int_0^1 f'(x)\frac{-1}{2\pi in}\mathrm{e}^{-2\pi inx}\,dx\right] = \frac{1}{2\pi in}\int_0^1 f'(x)\mathrm{e}^{-2\pi inx}\,dx.$$

More generally by integrating by parts k times

$$\widehat{f}(n) = \frac{1}{(2\pi in)^k}\int_0^1 f^{(k)}(x)\mathrm{e}^{-2\pi inx}\,dx = \frac{1}{(2\pi in)^k}\widehat{f^{(k)}}(n).$$

Since $f \in C^k$, we may apply the Riemann-Lebesgue lemma to $f^{(k)}$ and we deduce that

$$\lim_{|n|\to\infty} |\widehat{f^{(k)}}(n)| = 0$$

and therefore

$$\lim_{|n|\to\infty} |n|^k\widehat{f}(n) = 0.$$

Exercise 3. Let f be 2π-periodic and integrable on $[-\pi,\pi]$. We use the notation $\hat{f}_n = c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}dx$.

(i) Show that

$$\widehat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)\mathrm{e}^{-inx}\,dx$$

and hence

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]\mathrm{e}^{-inx}\,dx.$$

(ii) Now assume that f satisfies a Hölder condition of order $0<\alpha<1$, namely that there exists $C>0$ such that

$$|f(x+h) - f(x)| \le C|h|^{\alpha} \quad \forall x, h \in \mathbb{R}\,.$$

Use (i) to show that

$$\widehat{f}(n) = O(1/|n|^{\alpha}).$$

(iii) Prove that the result cannot be improved by showing that for $0<\alpha<1$ fixed, the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}\mathrm{e}^{i2^k x}, \tag{3}$$

satisfies

$$|f(x+h) - f(x)| \le C|h|^{\alpha} \quad \forall h \in \mathbb{R}$$

and $\widehat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$.

Hint: For (iii), break up the sum as follows

$$f(x+h) - f(x) = \sum_{2^k \le 1/|h|} + \sum_{2^k > 1/|h|}\,.$$

To estimate the first sum use the fact that $|1-\mathrm{e}^{i\theta}| \le |\theta|$ whenever θ is small. To estimate the second sum, use the obvious inequality $|\mathrm{e}^{ix} - \mathrm{e}^{iy}| \le 2$.

Solution:

(i) We have, after a change of variable and using the periodicity of f that

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\mathrm{e}^{i\pi}\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-in\left(x-\frac{\pi}{n}\right)}\,dx \\ &= -\frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\frac{\pi}{n})\mathrm{e}^{-iny}\,dy = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n}\right)\mathrm{e}^{-inx}\,dx.\end{aligned}$$

Thus,

$$\widehat{f}(n) = \frac{1}{2}\left[\widehat{f}(n) + \widehat{f}(n)\right] = \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right)\mathrm{e}^{-inx}\,dx\right]$$
$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[f(x) - f(x - \pi/n)\right]\mathrm{e}^{-inx}\,dx.$$

(ii) With the rewriting of (i), we can use the Hölder condition of order α to estimate

$$|\widehat{f}(n)| \le \frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x) - f(x + \pi/n)|\,dx \le \frac{1}{4\pi}\int_{-\pi}^{\pi} C\frac{\pi^\alpha}{|n|^\alpha}\,dx = \frac{1}{2}C\frac{\pi^\alpha}{|n|^\alpha} \le \frac{C\pi^\alpha}{2}\frac{1}{|n|^\alpha}.$$

Thus, $\widehat{f}(n) = O(1/|n|^\alpha)$.

(iii) Fix $0 < \alpha < 1$ and consider f defined in (3). We easily see that $\widehat{f}(N) = 1/N^\alpha$ whenever $N = 2^k$. We are left to show that f satisfies a Hölder condition of order α. We write

$$f(x+h) - f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}(\mathrm{e}^{i2^k(x+h)} - \mathrm{e}^{i2^k x}) = \sum_{k=0}^{\infty} 2^{-k\alpha}\mathrm{e}^{i2^k x}(\mathrm{e}^{i2^k h} - 1)$$
$$= \sum_{2^k \le 1/|h|} 2^{-k\alpha}\mathrm{e}^{i2^k x}(\mathrm{e}^{i2^k h} - 1) + \sum_{2^k > 1/|h|} 2^{-k\alpha}\mathrm{e}^{i2^k x}(\mathrm{e}^{i2^k h} - 1).$$

More precisely, let j be the smallest integer such that $2^j > 1/|h|$, i.e. $2^{j-1} \le 1/|h|$. Then, using the geometric series,

$$|f(x+h) - f(x)| = \sum_{k=0}^{j-1} 2^{-k\alpha}|\mathrm{e}^{i2^k x}|\underbrace{|\mathrm{e}^{i2^k h} - 1|}_{\le 2^k|h|} + \sum_{k=j}^{\infty} 2^{-k\alpha}|\mathrm{e}^{i2^k x}|\underbrace{|\mathrm{e}^{i2^k h} - 1|}_{\le 2}$$
$$\le \sum_{k=0}^{j-1} 2^{-k\alpha}2^k|h| + 2\sum_{k=j}^{\infty} 2^{-k\alpha}$$
$$\le 2^{1-j}\underbrace{\sum_{k=0}^{j-1} 2^{(1-\alpha)k}}_{\frac{2^{j(1-\alpha)} - 1}{2^{1-\alpha} - 1}} + 2^{-j\alpha}\underbrace{2\sum_{k=0}^{\infty} 2^{-k\alpha}}_{=\frac{2}{1-2^{-\alpha}}}$$
$$= 2^{-j\alpha}\left(\frac{2}{2^{1-\alpha} - 1} + \frac{2^{1+\alpha}}{2^\alpha - 1}\right) \le C|h|\,,$$

where we set

$$C = C(\alpha) := \frac{2}{2^{1-\alpha} - 1} + \frac{2^{1+\alpha}}{2^\alpha - 1} > 0\,.$$

Since C is independent of h and x this proves that f satisfies a Hölder condition of order α.

Exercise 4.

(i) Compute the Fourier series of the 2π-periodic odd function f defined by $f(x) = x(\pi - x)$ on $[0, \pi]$.

(ii) Using (i) and Parseval's identity, deduce the value of the series

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}.$$

Solution:

(i) First, since f is odd, $a_n = 0$ for all $n \geq 0$. Integrating by parts several times, we get

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\left(\underbrace{\left[-\frac{1}{n}x(\pi - x)\cos(nx)\right]_0^{\pi}}_{=0} + \int_0^{\pi} \frac{1}{n}(\pi - 2x)\cos(nx)\,dx\right) \\
&= \frac{2}{\pi}\left(\underbrace{\left[\frac{1}{n^2}(\pi - 2x)\sin(nx)\right]_0^{\pi}}_{=0} + \int_0^{\pi} \frac{2}{n^2}\sin(nx)\,dx\right) \\
&= \frac{2}{\pi}\left[-\frac{2}{n^3}\cos(nx)\right]_0^{\pi} = \frac{2}{\pi}\left(-\frac{2}{n^3}(-1)^n + \frac{2}{n^3}\right).
\end{aligned}$$

Thus,

$$b_n = \frac{4}{\pi n^3}\left(1 - (-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{8}{\pi n^3} & \text{if } n \text{ is odd .} \end{cases}$$

The Fourier series is therefore given by

$$Ff(x) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2k-1)^3}\sin((2k-1)x).$$

(ii) From Parseval's identity and the fact that f^2 is even, we deduce that

$$\begin{aligned}
\frac{64}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} &= \sum_{k=1}^{\infty} b_k^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{2}{\pi}\int_0^{\pi}(x(\pi - x))^2\,dx \\
&= \frac{2}{\pi}\int_0^{\pi}(\pi^2x^2 - 2\pi x^3 + x^4)\,dx = \frac{1}{15}\pi^4.
\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} = \frac{\pi^6}{960}.$$

Exercise 5. Let $f \in C^0([0,1])$ be a 1-periodic function and τ an irrational number. Show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\tau) = \int_0^1 f(x)\, dx$$

and that this result does not hold when τ is rational.
Hint: Begin by showing the result for functions of the form $f(x) = \mathrm{e}^{2\pi i k x}$ for some $k \in \mathbb{Z}$ and conclude by approximation.

Solution: As a counterexample with τ rational we can take

$$f(x) = \sin^2(2\pi x) \quad \text{and} \quad \tau = 1.$$

Now we prove the result in several steps.
Step 1: We begin by showing the result for functions of the form $f(x) = \mathrm{e}^{2\pi i k x}$ for $k \in \mathbb{Z}$.
If $f = 1$ the result is trivial. With $f(x) = \mathrm{e}^{2\pi i k x}$ where $k \neq 0$, we have

$$\int_0^1 \mathrm{e}^{2\pi i k x}\, dx = 0$$

and since τ is irrational $\mathrm{e}^{2\pi i k \tau} \neq 1$ so that by the geometric series

$$\frac{1}{N}\sum_{n=1}^{N} f(n\tau) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k n\tau} = \frac{1}{N}\left(\frac{1-e^{2\pi i k\tau(N+1)}}{1-e^{2\pi i k\tau}} - 1\right) = \frac{\mathrm{e}^{2\pi i k\tau}}{N}\frac{1-\mathrm{e}^{2\pi i k N\tau}}{1-\mathrm{e}^{2\pi i k\tau}}.$$

Since

$$\left|\frac{\mathrm{e}^{2\pi i k\tau}}{N}\right| \leq \frac{1}{N} \quad \text{and} \quad \left|\frac{1-\mathrm{e}^{2\pi i k N\tau}}{1-\mathrm{e}^{2\pi i k\tau}}\right| \leq \frac{2}{|1-\mathrm{e}^{2\pi i k\tau}|}$$

we deduce that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\tau) = 0.$$

Step 2: We now show the result for trigonometric polynomials.
Let f be a trigonometric polynomial, that is f is of the form

$$f(x) = \sum_{|k|\leq M} a_k \mathrm{e}^{2\pi i k x} \quad \text{where} \quad a_k \in \mathbb{C} \text{ for all } k.$$

That the result holds for f is an immediate consequence of the previous step and linearity of the sum and integral. In other words, using the previous Step, we have

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(n\tau) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{|k|\leq M}a_k\mathrm{e}^{2\pi ikx}\\ &= \sum_{|k|\leq M}a_k\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}^{2\pi ikx}\\ &= \sum_{|k|\leq M}a_k\int_0^1\mathrm{e}^{2\pi ikx}\,dx\\ &= \int_0^1\sum_{|k|\leq M}a_k\mathrm{e}^{2\pi ikx}\,dx\\ &= \int_0^1 f(x)\,dx.\end{aligned}$$

Step 3: We now show the result for general continuous functions.
From the Weierstrass' Theorem, we know that the trigonometric polynomials are dense in the space $C^0([0,1])$, so for any $f\in C^0([0,1])$ and $\varepsilon>0$ arbitrary, we know that there exists a trigonometric polynomial P such that $\|f-P\|_{C^0}<\varepsilon$. As a consequence, we have thta

$$\left|\int_0^1 f(x)\,dx-\int_0^1 P(x)\,dx\right|<\varepsilon$$

and that for all $n\geq 1$ $|f(n\tau)-P(n\tau)|<\varepsilon$. The latter implies that for all $N\geq 1$, we have

$$\left|\frac{1}{N}\sum_{k=1}^{N}f(n\tau)-\frac{1}{N}\sum_{k=1}^{N}P(n\tau)\right|<\varepsilon$$

Since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P(n\tau)=\int_0^1 P(x)\,dx,$$

there is $N_\varepsilon\geq 0$ such that for all $N\geq N_\varepsilon$

$$\left|\frac{1}{N}\sum_{n=1}^{N}P(n\tau)-\int_0^1 P(x)\,dx\right|<\varepsilon.$$

Finally, we deduce that for all $N\geq N_\varepsilon$

$$\begin{aligned}&\left|\frac{1}{N}\sum_{k=1}^{N}f(n\tau)-\int_0^1 f(x)\,dx\right|\\ &\leq\left|\frac{1}{N}\sum_{k=1}^{N}f(n\tau)-\frac{1}{N}\sum_{k=1}^{N}P(n\tau)\right|+\left|\frac{1}{N}\sum_{n=1}^{N}P(n\tau)-\int_0^1 P(x)\,dx\right|+\left|\int_0^1 f(x)\,dx-\int_0^1 P(x)\,dx\right|<3\varepsilon.\end{aligned}$$

Since ε was arbitrary, we deduce that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}f(n\tau)=\int_0^1 f(x)\,dx\,.$$

Exercise 6 ($\star$)**.** Construct a continuous function whose Fourier series diverges at some point.

Hint: You may work on $[-\pi,\pi]$. For $N\in\mathbb{N}$, we define $\psi_N(x):=\operatorname{sgn}(x)\sin((N+\frac{1}{2})x)$ and choose the ansatz $f(x):=\sum_{k=1}^\infty a_k\psi_{N_k}(x)$ for a sequence $\{a_k\}_{k=1}^\infty$ of positive real numbers (what assumption on $\{a_k\}$ do you need?) and sequence $\{N_k\}_{k=1}^\infty$ of integers. Choose the sequence N_k carefully, depending on $\{a_k\}_k$, such that the Fourier series diverges in some point.

Solution: Let $\{a_k\}$ be a sequence of positive real numbers with $\sum_{k=1}^\infty a_k<+\infty$. For $N\in\mathbb{N}$ we define $\psi_N(x):=\operatorname{sgn}(x)\sin((N+\frac{1}{2})x)$. We make the ansatz

$$f(x)=\sum_{k=1}^\infty a_k\psi_{N_k}(x)$$

for some sequence $\{N_k\}_{k=1}^\infty$ in $\mathbb{N}$ yet to be chosen. Observe that since ψ_{N_k} is bounded by 1 and continuous for every $N_k\in\mathbb{N}$ and since $\sum_{k=1}^\infty a_k<\infty$, we have that $\lim_{N\to\infty}\sum_{k=1}^N a_k\psi_{N_k}(x)$ converges uniformly. In particular, it follows that f is continuous. We know claim that there exists a choice of the sequence $\{N_k\}_{k=1}^\infty$ such that the Fourier series diverges in 0. More precisely, we will prove that with an appropriate choice of $\{N_k\}_{k=1}^\infty$, that

$$\lim_{N\to\infty}S_Nf(0)=+\infty\,,$$

where $S_Nf(x):=\sum_{|n|\le N}\hat{f}(n)e^{inx}$ denotes the Nth partial Fourier sum. Recall from the lecture (or observe) that that for a continuous, 2π-periodic function f, we have that

$$S_Nf(x)=\int_{-\pi}^{\pi}D_N(x-y)f(y)\,dy \tag{4}$$

where $D_N(x):=\frac{\sin((N+\frac{1}{2})x)}{2\pi\sin(\frac{x}{2})}$ is the Dirichlet kernel. Indeed, (4) just follows from observing that from the geometric series

$$\begin{aligned}\sum_{|n|\le N}e^{inx}&=\sum_{n=0}^N e^{inx}+\sum_{n=1}^N e^{-inx}=\frac{1-e^{i(N+1)x}}{1-e^{ix}}+\frac{1-e^{-i(N+1)x}}{1-e^{-ix}}-1=\frac{e^{-iNx}-e^{i(N+1)x}}{1-e^{ix}}\\&=\frac{e^{-i(N+\frac{1}{2})x}}{e^{-ix/2}-e^{ix/2}}=\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}\,.\end{aligned}$$

Hence it suffices to prove that with an appropriate choice of $\{N_k\}_{k=1}^\infty$ (notice that $D_N(-y)=D_N(y)$)

$$\lim_{N\to\infty}S_Nf(0)=\lim_{N\to\infty}\int_{-\pi}^{\pi}D_N(y)f(y)\,dy=\lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin((N+\frac{1}{2})y)}{2\pi\sin(\frac{y}{2})}f(y)\,dy=+\infty\,. \tag{5}$$

Step 1: We prove that for all $N\in\mathbb{N}$, we have the lower bound

$$\int_{-\pi}^{\pi}D_N(y)\psi_N(y)\,dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2[(N+\frac{1}{2})y]}{|\sin(\frac{y}{2})|}\,dy\ge\frac{\log(N+1)}{\pi}\,. \tag{6}$$

Observe that by definition and since the integrand is even, we have

$$\int_{-\pi}^{\pi} D_N(y)\psi_N(y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin^2[(N+\frac{1}{2})y]}{|\sin(\frac{y}{2})|}\,dy = \frac{1}{\pi}\int_0^{\pi} \frac{\sin^2[(N+\frac{1}{2})y]}{\sin(\frac{y}{2})}\,dy\,.$$

Notice that $\sin(\frac{y}{2}) \le \frac{y}{2}$ for $y \in [0,\pi]$, so that we can estimate from below

$$\begin{aligned}\int_0^{\pi} \frac{\sin^2[(N+\frac{1}{2})x]}{|\sin(\frac{x}{2})|}\,dx &\ge 2\int_0^{\pi} \frac{\sin^2[(N+\frac{1}{2})x]}{x}\,dx \\ &= 2\int_0^{(N+\frac{1}{2})\pi} \frac{\sin^2(y)}{y}\,dy \\ &= \sum_{l=1}^{N} 2\int_{(l-1)\pi}^{l\pi} \frac{\sin^2(y)}{y}\,dy + \int_{N\pi}^{(N+\frac{1}{2})\pi} \frac{\sin^2(y)}{y}\,dy.\end{aligned}$$

Notice that

$$\int_{(l-1)\pi}^{l\pi} \frac{\sin^2(y)}{y}\,dy \ge \frac{1}{l\pi}\underbrace{\int_{(l-1)\pi}^{l\pi} \sin^2(y)\,dy}_{=\pi/2} = \frac{1}{2l}\,.$$

Thus,

$$\int_0^{\pi} \frac{\sin^2[(N+\frac{1}{2})x]}{\sin(\frac{x}{2})}\,dx \ge \sum_{l=1}^{N}\frac{1}{l} = \sum_{l=1}^{N}\int_l^{l+1}\frac{1}{l}\,dy \ge \sum_{l=1}^{N}\int_l^{l+1}\frac{dy}{y} = \int_1^{N+1}\frac{dy}{y} = \log(N+1).$$

We conclude (6).

Step 2: We prove that for all $N \in \mathbb{N}$ and $\varepsilon > 0$, there is $M_0 \in \mathbb{N}$ such that for all $M \ge M_0$

$$\left|\int_{-\pi}^{\pi} D_M(y)\psi_N(y)\,dy\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin[(M+\frac{1}{2})y]}{\sin(\frac{y}{2})}\psi_N(y)\,dy\right| \le \varepsilon.$$

The claim follows from the Riemann-Lebesgue lemma if we manage to prove that

$$x \mapsto \frac{\psi_N(x)}{\sin(\frac{x}{2})}$$

is a function in $L^1(-\pi,\pi)$. It is sufficient to show that this function is bounded. Note that it is continuous on $(-\infty,0)\cup(0,+\infty)$ and using De L'Hopital's rule we can show that the function is bounded by $2N+1$. Thus, the claim follows.

Step 3: We recursively choose the sequence $\{N_k\}_{k=1}^{\infty}$ such that (5) holds.

Let us select the sequence $\{N_k\}_{k=1}^{\infty}$ recursively. Indeed, assume that we've already selected $N_1,\dots,N_{k-1}$. We then choose N_k such that

(i) $a_k\dfrac{\log(N_k+1)}{2} > k$,

(ii) for all $l < k$, it holds that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin[(N_k+\frac{1}{2})x]}{\sin(\frac{x}{2})}\psi_{N_l}\,dx\right| \le 1.$$

The fact that we can select an $N_k \in \mathbb{N}$ satisfying (i) and (ii) follows from Step 2. Moreover, with this choice we have, recalling (4), that

$$\begin{aligned}
S_{N_j} f(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} f(x)\, dx \\
&= \sum_{k=1}^{j-1} a_k \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_k}(x)\, dx \\
&+ a_j \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_j}(x)\, dx \\
&+ \sum_{k=j+1}^{\infty} a_k \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_k}(x)\, dx.
\end{aligned}$$

Note that the permutation of the sum and the integral above was permitted because of the uniform convergence. Now we have

- for $k < j$ due to the requirement (ii) that

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_k}(x)\, dx \right| \leq 1,$$

- for $k = j$ due to Step 1 that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_j}(x)\, dx \geq \frac{\log(N_j + 1)}{\pi};$$

- for $k > j$ again due to requirement (ii) that

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_j + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_k}(x)\, dx \right| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin[(N_k + \frac{1}{2})x]}{\sin(\frac{x}{2})} \psi_{N_j}(x)\, dx \right| \leq 1.$$

Thus, finally, we get using (i) that

$$S_{N_j} f(0) \geq -\sum_{k=1}^{j-1} a_k + a_j \frac{\log(N_j + 1)}{\pi} - \sum_{k=j+1}^{\infty} a_k \geq \frac{j}{\pi} - \sum_{k=1}^{\infty} a_k \to +\infty$$

as $j \to \infty$ as desired.