Serie 10
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. We want to generalise the results for 1-periodic functions from the lecture to periodic functions with period not necessarily equal to 1 . To this end, let $L > 0$ and let $f\colon \mathbb{R} \to \mathbb{C}$ be a complex-valued, continuous and L-periodic function. For $n \in \mathbb{Z}$, we define

$$c_n = \frac{1}{L}\int_0^L f(x)\mathrm{e}^{-\frac{2\pi i n x}{L}}\,dx.$$

(i) Show that the series

$$\sum_{-\infty}^{\infty} c_n \mathrm{e}^{\frac{2\pi i n x}{L}}$$

converges to f in $L^2(0,L)$. More precisely, show that

$$\lim_{N\to\infty}\int_0^L \left| f(x) - \sum_{n=-N}^{N} c_n \mathrm{e}^{\frac{2\pi i n x}{L}}\right|^2 dx = 0.$$

(ii) If the series $\sum_{n=-\infty}^{+\infty}|c_n|$ is absolutely convergent, show that

$$\sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{\frac{2\pi i n x}{L}}$$

converges uniformly to f.

(iii) Show that

$$\frac{1}{L}\int_0^L |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

Hint: For (iii), apply Parseval's identity to the function $x \mapsto f(Lx)$.

Solution:

(i) Consider the function $g\colon \mathbb{R} \to \mathbb{C}$ given by $g(x) = f(Lx)$ which is 1-periodic. Let $\{a_n\}_{n=-\infty}^{\infty}$ be the Fourier coeffcients of g. From the Fourier theorem we know that

$$\lim_{N\to\infty} \int_0^1 \left| g(x) - \sum_{n=-N}^{N} a_n \mathrm{e}^{2\pi i n x} \right|^2 dx = 0.$$

Notice that

$$a_n = \int_0^1 g(x)\mathrm{e}^{2\pi i n x}\, dx = \frac{1}{L}\int_0^L f(x)\mathrm{e}^{\frac{2\pi i n x}{L}}\, dx = c_n.$$

Thus,

$$\lim_{N\to\infty} \int_0^1 \left| f(Lx) - \sum_{n=-N}^{N} c_n \mathrm{e}^{2\pi i n x} \right|^2 dx = 0$$

which implies, with the change of variables $y := Lx$, that

$$\lim_{N\to\infty} \int_0^L \left| f(x) - \sum_{n=-N}^{N} c_n \mathrm{e}^{\frac{2\pi i n x}{L}} \right|^2 dx = 0.$$

(ii) Consider the Nth partial Fourier sum $S_N f\colon \mathbb{R} \to \mathbb{C}$ given by

$$S_N f(x) = \sum_{n=-N}^{N} c_n \mathrm{e}^{\frac{2\pi i n x}{L}}.$$

We will prove that $\{S_N f\}_{N=1}^{\infty}$ is a Cauchy sequence in $(C(\mathbb{R}), \|\cdot\|_\infty)$. By the absolute convergence of $\{c_n\}_{n\in\mathbb{Z}}$, there exists for any $\varepsilon > 0$ a large M such that $\sum_{|n|\geq M} |c_n| < \varepsilon$. Then, for any $n \geq m \geq M$,

$$|S_n f(x) - S_m f(x)| = \left| \sum_{m<|k|\leq n} c_k \mathrm{e}^{\frac{2\pi i k x}{L}} \right| \leq \sum_{m<|k|} |c_k| < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

Thus $\{S_N f\}_{N=1}^{\infty}$ is a Cauchy sequence in $(C(\mathbb{R}), \|\cdot\|_\infty)$ and therefore it converges to some function $g \in C(\mathbb{R})$. Since $S_N f \to g$ uniformly, $S_N f \to g$ in $L^2(0, L)$ as well. Thus, $f = g$ in $L^2(0, L)$, which proves that they agree a.e. but since they are both continuous $f = g$ everywhere (see Exercise sheet 8) and hence the series converges uniformly to f.

(iii) Let $\{a_n\}_{n=-\infty}^{\infty}$ be as in (i). Using Parseval's identity,

$$\int_0^1 |f(Lx)|^2\, dx = \int_0^1 |g(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |a_n|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

We conclude after the change of variables $y := Lx$ that

$$\frac{1}{L}\int_0^L |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Exercise 2. Fourier series sometimes yield an elegant way to compute the value of a series.

(i) Compute the Fourier series of the function defined by $f(x) = (2x-1)^2$ on $[0, 1[$ and extended to a 1-periodic function on $\mathbb{R}$.

(ii) Compare f and its Fourier series Ff on $[0,1]$.

(iii) Use (ii) to compute the value of the convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution:

(i) We compute the Fourier coefficients. Observe that f is even with respect to $\frac{1}{2}$ hence $b_n = 0$, for all $n \geq 1$. As for the other coefficients, we have

$$a_0 = 2\int_0^1 (2x-1)^2\,dx = \int_{-1}^1 u^2\,du = \frac{2}{3}.$$

Integrating by parts twice, we get

$$\begin{aligned} a_n &= 2\int_0^1 (2x-1)^2 \cos(2\pi n x)\,dx \\ &= 2\Bigg(\underbrace{\left[\frac{1}{2\pi n}(2x-1)^2 \sin(2\pi n x)\right]_0^1}_{=0} - \int_0^1 \frac{2}{\pi n}(2x-1)\sin(2\pi n x)\,dx\Bigg) \\ &= -\frac{4}{\pi n}\int_0^1 (2x-1)\sin(2\pi n x)\,dx \\ &= \frac{4}{\pi n}\Bigg(\left[\frac{1}{2\pi n}(2x-1)\cos(2\pi n x)\right]_0^1 - \underbrace{\int_0^{2\pi} \frac{1}{\pi n}\cos(2\pi n x)\,dx}_{=0}\Bigg) \\ &= \frac{4}{(\pi n)^2}. \end{aligned}$$

Thus, the Fourier series of f is given by

$$Ff(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{(\pi n)^2}\cos(2\pi n x).$$

(ii) Since $f \in C^{0,1}([0,1])$, the Dirichlet theorem gives that the Fourier series converges pointwise for any $x \in [0,1]$. Hence we have for all $x \in [0,1]$

$$f(x) = Ff(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{(\pi n)^2}\cos(2\pi n x). \tag{1}$$

(iii) Evaluating (1) in $x = \frac{1}{2}$, we get

$$0 = f\Big(\frac{1}{2}\Big) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{(\pi n)^2}(-1)^n,$$

which implies

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Again, evaluating (1) in $x = 0$, we get

$$1 = f(0) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{(\pi n)^2},$$

which implies

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 3. We have seen conditions that ensure the pointwise convergence of the Fourier series - but does it converge also absolutely? In general, this is not the case, as the following example illustrates. Let $[a, b] \subseteq [0, 1]$ and consider the indicator function $f(x) = \mathbb{1}_{[a,b]}(x)$.

(i) Compute the Fourier series of f.

(ii) If $a \neq 0$, $b \neq 1$ and $a \neq b$, show that the Fourier series doesn't converge absolutely; however, it converges pointwise for every x.

(iii) What happens if $a = 0$ and $b = 1$?

Solution:

(i) We begin by computing the Fourier coefficients:

$$c_0 = \int_0^1 f(x)\, dx = \int_a^b dx = b - a$$

$$c_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx = \int_a^b e^{-inx}\, dx = \left[\frac{-1}{2\pi i n} e^{-2\pi i n x}\right]\Bigg|_a^b = \frac{1}{2\pi i n}(e^{-2\pi i n a} - e^{-2\pi i n b}).$$

Thus, the Fourier series is given by

$$Ff(x) = b - a + \sum_{n \neq 0} \frac{e^{-2\pi i n a} - e^{-2\pi i n b}}{2\pi i n} e^{2\pi i n x}.$$

(ii) Notice that

$$e^{-2\pi i n a} - e^{-2\pi i n b} = e^{2\pi i n \frac{b+a}{2}} (e^{2\pi i n \frac{b-a}{2}} - e^{2\pi i n \frac{a-b}{2}}) = 2i e^{2\pi i n \frac{b+a}{2}} \sin(\pi n (b - a)).$$

Thus,

$$|c_n| = \left| \frac{e^{-2\pi i n a} - e^{-2\pi i n b}}{2\pi i n} \right| = \frac{1}{\pi n} |\sin(\pi n (b - a))| = \frac{1}{\pi n} |\sin(n\theta)|$$

where $\theta := \pi(b-a)$. Since $\theta < \pi$, there exists N large enough such that for all $k \geq 1$, there exists $n \in [Nk+1, N(k+1)]$ such that $|\sin(n\theta)| > 1/2$. Thus,

$$\sum_{n\in\mathbb{Z}} |c_n| = \infty,$$

and therefore the series does not converge absolutely. Alternatively, it is enough to recall from the lecture that absolute convergence of the series $\sum_{n\in\mathbb{Z}} c_n$ would imply that the Fourier series converges to f uniformly. The latter is however impossible as f is not continuous.

Finally, we show that the series converges pointwise. Indeed, the quantities

$$\lim_{\substack{x > x_0 \\ \to}} f(x) \quad \text{and} \quad \lim_{\substack{x < x_0 \\ \to}} f(x)$$

are well-defined for every x_0. Thus, since f is continuously differentiable everywhere except in a and b, it follows from Dirichlet's theorem that the Fourier series converges pointwise. More precisely, we have that pointwise

$$Ff(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ 0 & \text{if } x \in [0,a) \cup (b,1] \\ \frac{1}{2} & \text{if } x \in \{a,b\}\,. \end{cases}$$

(iii) If $a = 0$, $b = 1$, $c_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, so that for every $x \in [0,1]$

$$f(x) = \mathbb{1}_{[0,1]}(x) \equiv 1 = c_0.$$

Exercise 4. We consider the so-called sawtooth function, that is the 2π-periodic function f defined by

$$f(x) = \begin{cases} -i(\pi + x) & -\pi < x < 0, \\ 0 & x = 0, \\ i(\pi - x) & 0 < x < \pi. \end{cases} \tag{2}$$

(i) Recall Exercise 1 and compute the Fourier series Ff.

(ii) Compare Ff and f.

(iii) Can we differentiate term by term the Fourier series, i.e is the derivative of f equal to the sum of the derivaties of every term in the Fourier series?

Solution:

(i) Observe that from Exercise 1, we know that the Fourier coefficients of a 2π-periodic function

$f : \mathbb{R} \to \mathbb{C}$ are given by, for $n \in \mathbb{Z}$,

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} f(x)e^{-inx}\,dx + \frac{1}{2\pi}\int_{-\pi}^{0} f(y+2\pi)e^{-in(y+2\pi)}\,dy$$
$$= \frac{1}{2\pi}\int_0^{\pi} f(x)e^{-inx}\,dx + \frac{1}{2\pi}\int_{-\pi}^{0} f(y)e^{-iny}\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\,,$$

where in the second equality, we used the change of variables $y := x - 2\pi$ and in the third, we used the 2π-periodicity of the integrand. As odd function, f has average 0 and hence

$$c_0 = 0,$$

and for $n \neq 0$,

$$c_n = -\frac{i}{2\pi}\int_{-\pi}^{0} (\pi + x)\mathrm{e}^{-inx}\,dx + \frac{i}{2\pi}\int_0^{\pi} (\pi - x)\mathrm{e}^{-inx}\,dx.$$

We compute these two integrals separately:

$$\int_{-\pi}^{0} (\pi + x)\mathrm{e}^{-inx}\,dx = (\pi + x)\frac{1}{-in}\mathrm{e}^{-inx}|_{-\pi}^{0} - \frac{1}{(-in)^2}\mathrm{e}^{-inx}|_{-\pi}^{0} = -\frac{\pi}{in} - \frac{2}{(in)^2},$$
$$\int_0^{\pi} (\pi - x)\mathrm{e}^{-inx}\,dx = (\pi - x)\frac{1}{(-in)}e^{-inx}|_0^{\pi} - \frac{-1}{(in)^2}\mathrm{e}^{-inx}|_0^{\pi} = \frac{\pi}{in} - \frac{2}{(in)^2}.$$

Thus,

$$c_n = \frac{1}{n}.$$

We conclude that the Fourier series of f is given by

$$Ff(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{inx}}{n} = 2i\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}\,.$$

(ii) Since f is piecewise continuous and continuously differentiable except in $x = 0$, we deduce from Dirichlet's theorem that pointwise

$$Ff(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ \frac{1}{2}(\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)) = 0 = f(0) & \text{if} x = 0\,, \end{cases}$$

hence $Ff(x) = f(x)$ pointwise everywhere.

(iii) We observe that $f'(x) = -i$ for all $x \neq 0$. If, however, we differentiate the Fourier series term by term, we would get

$$\frac{d}{dx}Ff(x) = 2i\sum_{n=1}^{\infty} \cos(nx)\,,$$

which doesn't converge since $\lim_{n\to\infty} \cos(nx) \neq 0$. Hence, we cannot differentiate the Fourier series term by term (and in fact, since f is not continuous in 0, f doesn't verify the criterion of the lecture which allows to differentiate the Fourier series term by term.)

Exercise 5 ($\star$)**.** We have seen, by means of the Fourier series, that trigonometric polynomials are dense in the space of continuous, periodic functions. This raises the question whether all trigonometric series are the Fourier series of some continuous and periodic function. To answer this question, let us consider again the 2π-periodic sawtooth function introduced in (2) and recall its Fourier series that you computed in Exercise 4.

We want to understand what happens if we break the symmetry between the frequencies e^{inx} and e^{-inx} which appear in the Fourier expansion. Consider therefore the series

$$\sum_{n=-\infty}^{-1} \frac{e^{inx}}{n}, \tag{3}$$

and prove that it is no longer the Fourier series of a bounded function. In particular, this series is an example of a trigonometric series which is not a Fourier series.

Hint: Argue by contradiction. Assume that (3) is the Fourier series of a bounded function f. Consider the Cesaro means $\Phi_N f(x) = \frac{1}{N}\sum_{j=0}^{N-1} S_j f(x)$ and recall their connection with the Fejer kernel.

Solution: Suppose by contradiction that (3) is the Fourier series of a bounded function f. Let C be the constant such that $|f| \leq C$ everywhere. The N-th partial Fourier sum is given by

$$S_N f(x) = \sum_{-N \leq n \leq -1} \frac{e^{inx}}{n}.$$

Now, if $S_N \to f$ pointwise as $N \to \infty$, then the Cesaro means given by

$$\Phi_N f(x) = \frac{1}{N}\sum_{j=0}^{N-1} S_j f(x)$$

would converge to $f(x)$ as well. Therefore it would be interesting to look at what happens with the Cesaro means in this specific case. Recall that the definition of the Fejer kernel (here in its 2π-periodic version) is given by

$$F_N(x) = \frac{1}{2\pi}\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N}\right) e^{inx}$$

and that we have seen in Exercise sheet 9 that

$$F_N(x) = \frac{1}{2\pi}\frac{1}{N}\left|\sum_{n=0}^{N-1} e^{inx}\right|^2,$$

which implies $F_N \geq 0$, $\int_{-\pi}^{\pi} F_N(x)\,dx = 1$ and $\Phi_N f(x) = (F_N * f)(x)$ for all $x \in [-\pi, \pi]$. Thus,

$$|\Phi_N f(x)| \leq \int_{-\pi}^{\pi} |F_N(x-y) f(y)|\,dy \leq C \int_{-\pi}^{\pi} |F_N(x-y)|\,dy = C. \tag{4}$$

In this case however, we can compute Cesaro mean explicitly:

$$\Phi_N f(x) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{-n \leq k \leq -1} \frac{e^{ikx}}{k} = -\frac{1}{N}\sum_{n=1}^{N-1}(N-n)\frac{e^{-inx}}{n}.$$

Evaluating in $x = 0$, we get

$$\Phi_N f(0) = -\frac{1}{N}\sum_{n=1}^{N-1}\frac{N-n}{n} = -\frac{1}{N}\left[\sum_{n=1}^{N-1}\frac{N}{n} - (N-1)\right] \leq 1 - \sum_{n=0}^{N-1}\frac{1}{n}$$

so that

$$\lim_{N\to\infty}\Phi_N f(0) = -\infty,$$

but this contradicts (4). Thus, $\sum_{n=-\infty}^{-1}\frac{\mathrm{e}^{inx}}{n}$ is not a Fourier series of a bounded function.