Serie 10

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Monday at 8am via moodle. They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. We want to generalise the results for 1-periodic functions from the lecture to periodic functions with period not necessarily equal to 1. To this end, let $L > 0$ and let $f\colon \mathbb{R} \to \mathbb{C}$ be a complex-valued, continuous and L-periodic function. For $n \in \mathbb{Z}$, we define

$$c_n = \frac{1}{L} \int_0^L f(x) \mathrm{e}^{-\frac{2\pi i n x}{L}} \, dx.$$

(i) Show that the series

$$\sum_{-\infty}^{\infty} c_n \mathrm{e}^{\frac{2\pi i n x}{L}}$$

converges to f in $L^2(0, L)$. More precisely, show that

$$\lim_{N\to\infty} \int_0^L \left| f(x) - \sum_{n=-N}^{N} c_n \mathrm{e}^{\frac{2\pi i n x}{L}} \right|^2 dx = 0.$$

(ii) If the series $\sum_{n=-\infty}^{+\infty} |c_n|$ is absolutely convergent, show that

$$\sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{\frac{2\pi i n x}{L}}$$

converges uniformly to f.

(iii) Show that

$$\frac{1}{L} \int_0^L |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Hint: For (iii), apply Parseval's identity to the function $x \mapsto f(Lx)$.

Exercise 2. Fourier series sometimes yield an elegant way to compute the value of a series.

(i) Compute the Fourier series of the function defined by $f(x) = (2x-1)^2$ on $[0, 1[$ and extended to a 1-periodic function on $\mathbb{R}$.

(ii) Compare f and its Fourier series Ff on $[0,1]$.

(iii) Use (ii) to compute the value of the convergent series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Exercise 3. We have seen conditions that ensure the pointwise convergence of the Fourier series - but does it converge also absolutely? In general, this is not the case, as the following example illustrates. Let $[a,b] \subseteq [0,1]$ and consider the indicator function $f(x) = \mathbb{1}_{[a,b]}(x)$.

(i) Compute the Fourier series of f.

(ii) If $a \neq 0$, $b \neq 1$ and $a \neq b$, show that the Fourier series doesn't converge absolutely; however, it converges pointwise for every x.

(iii) What happens if $a = 0$ and $b = 1$?

Exercise 4. We consider the so-called sawtooth function, that is the 2π-periodic function f defined by

$$f(x) = \begin{cases} -i(\pi + x) & -\pi < x < 0, \\ 0 & x = 0, \\ i(\pi - x) & 0 < x < \pi. \end{cases} \tag{2}$$

(i) Recall Exercise 1 and compute the Fourier series Ff.

(ii) Compare Ff and f.

(iii) Can we differentiate term by term the Fourier series, i.e is the derivative of f equal to the sum of the derivaties of every term in the Fourier series?

Exercise 5 ($\star$)**.** We have seen, by means of the Fourier series, that trigonometric polynomials are dense in the space of continuous, periodic functions. This raises the question whether all trigonometric series are the Fourier series of some continuous and periodic function. To answer this question, let us consider again the 2π-periodic sawtooth function introduced in (2) and recall its Fourier series that you computed in Exercise 4.

We want to understand what happens if we break the symmetry between the frequencies e^{inx} and e^{-inx} which appear in the Fourier expansion. Consider therefore the series

$$\sum_{n=-\infty}^{-1} \frac{\mathrm{e}^{inx}}{n}, \tag{3}$$

and prove that it is no longer the Fourier series of a bounded function. In particular, this series is an example of a trigonometric series which is not a Fourier series.

Hint: Argue by contradiction. Assume that (3) is the Fourier series of a bounded function f. Consider the Cesaro means $\Phi_N f(x) = \frac{1}{N} \sum_{j=0}^{N-1} S_j f(x)$ and recall their connection with the Fejer kernel.