EPFL

Prof. : Maria Colombo
Analysis IV - XXX
01.07.2022
Duration : 180 minutes

98

XXX-5

SCIPER : **FAKE-5**

Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 24 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A4 paper with notes handwritten by you personally.**
- A table of Fourier transform pairs is provided on the last page of the document.
- For **multiple choice** questions, each question has **exactly one** correct answer. We give :

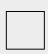
  +1 point if your answer is correct,
  0 points if you give no answer or more than one answer, or if your answer is incorrect.

- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

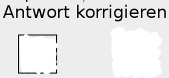
Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

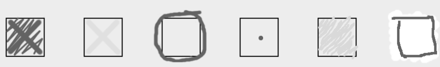
ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen



Corriger une réponse | Correct an answer
Antwort korrigieren



ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte



Part 1: multiple choice questions (10 points)

- For each question, mark the box corresponding to the correct answer.
- Each question has **exactly one** correct answer.
- For each multiple choice question, we give :

  +1 point if your answer is correct,
  0 points if you give no answer or more than one answer, or if your answer is incorrect.

Question 1 : Let $A = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3 \leq 1\}$, and define $f(x) = 1$ for all $x \in \mathbb{R}^3$. What is the value of $\int_A f\,dx$?

- ☐ 1/24.
- ☐ 1/6.
- ☐ 1.
- ☐ 1/2.

Question 2 : Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function. Define for $N = 1, 2, \cdots$ and $x \in \mathbb{R}$

$$S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

Which of the following statements is false ?

- ☐ If $f(x) = |\sin(x)|$ for $x \in \mathbb{R}$, then S_N converges to f uniformly on $\mathbb{R}$.
- ☐ There exists a (2π-periodic) function f which is Lipschitz continuous, but such that S_N does not converge to f pointwise as $N \to +\infty$.
- ☐ If f belongs to the space $L^2((0, 2\pi))$ then the sequence $(S_N)_N$ admits an almost everywhere convergent subsequence.
- ☐ If f is not continuous on $\mathbb{R}$, then the convergence of S_N to f cannot be uniform on $\mathbb{R}$.

Question 3 : Let $\Omega = (0, 1)$. For $x \in (0, 1)$, define $f(x) = x$. What can you infer from the Minkowski inequality ?

- ☐ $\|f + g\|_{L^2(\Omega)} \geq \frac{1}{3} + \|g\|_{L^2(\Omega)}$ for all $g \in L^2(\Omega)$.
- ☐ $\|f + g\|_{L^2(\Omega)} \leq \frac{1}{3} + \|g\|_{L^2(\Omega)}$ for all $g \in L^2(\Omega)$.
- ☐ $\|f + g\|^2_{L^2(\Omega)} \leq \frac{1}{3} + \|g\|^2_{L^2(\Omega)}$ for all $g \in L^2(\Omega)$.
- ☐ $\|f + g\|_{L^2(\Omega)} \leq \frac{\sqrt{3}}{3} + \|g\|_{L^2(\Omega)}$ for all $g \in L^2(\Omega)$.

Question 4 : Which of the following statements is true?

- ☐ If $f, g \colon \mathbb{R} \to \mathbb{R}$ are simple functions and $f(x) > 0$ for all $x \in \mathbb{R}$, then f^g is a simple function.
- ☐ If $f \colon \mathbb{R} \to \mathbb{R}$ is a simple function, then there exists a finite number of reals numbers $c_1, \ldots, c_N$ and a finite number of disjoint intervals $E_1, \ldots, E_N$ for which $f(x) = \sum_{i=1}^{N} c_i \mathbf{1}_{E_i}(x)$ for all $x \in \mathbb{R}$.
- ☐ There exists a simple function $g \colon \mathbb{R} \to \mathbb{R}$ and a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is not simple.
- ☐ If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function and a pointwise limit of simple functions, then f is simple.

Question 5 : Let $f : \mathbb{R} \to \mathbb{C}$ be a 1-periodic function such that f can be written as a pointwise limit of Fourier series whose coefficients $\hat{f}(n)$ for $n \in \mathbb{Z}$ are given by

$$\hat{f}(n) = \begin{cases} 2^{-n} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Then the function $f : \mathbb{R} \to \mathbb{C}$ is given by

- ☐ $f(x) = \frac{4-2\cos(2\pi x)-2i\sin(2\pi x)}{5-4i\sin(2\pi x)} - 1.$
- ☐ $f(x) = \frac{4-2\cos(2\pi x)-2i\sin(2\pi x)}{5-4\cos(2\pi x)} - 1.$
- ☐ $f(x) = \frac{4-2\cos(2\pi x)+2i\sin(2\pi x)}{5-4i\sin(2\pi x)}.$
- ☐ $f(x) = \frac{4-2\cos(2\pi x)+2i\sin(2\pi x)}{5-4\cos(2\pi x)}.$

Question 6 : For which of the following sequences of measurable functions $f_n : \Omega \to \mathbb{R}$, with $\Omega \subseteq \mathbb{R}$ measurable, is it true that

$$\lim_{n\to\infty} \int_\Omega f_n(x)\, dx = 0\,? \tag{1}$$

- ☐ $f_n(x) := (x^n - 1)\log(x)$ and $\Omega := (0,1)\,.$
- ☐ $f_n(x) := \frac{x^2+nx}{n}$ and $\Omega := (0,1)\,.$
- ☐ $f_n(x) := \frac{x^n-1}{\log(x)}$ and $\Omega := (0,1)\,.$
- ☐ $f_n(x) := e^{-nx^2}$ and $\Omega := (0,\infty)\,.$

Question 7 : Which of the following statements is true?

- ☐ If A is a dense subset of $\mathbb{R}$, then either $m^*(A) = 0$ or $m^*(A) = \infty$.
- ☐ If $f\colon \mathbb{R} \to \mathbb{R}$ is measurable and $A \subseteq \mathbb{R}$ is measurable, then $f(A)$ is measurable.
- ☐ If $A \subseteq \mathbb{R}$ is a non-measurable set and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, then $f(A)$ is not measurable.
- ☐ If C is a subset of $\mathbb{R}^n$ such that $m^*(A \cup (B \cap C)) + m^*(A \cap B \cap C) = m^*(A) + m^*(B \cap C)$ for all sets $A, B \subseteq \mathbb{R}^n$, then C is measurable.

Question 8 : Which of the following statements is true?

- ☐ There is a measurable set $A \subseteq \mathbb{R}$ with $m(A) = 0$ such that $\int_\mathbb{R} I_A(x)dx \neq \int_\mathbb{R} \mathbf{1}_A(x)dx$. Here, $\mathbf{1}_A$ is the indicator function on A, and I_A is defined as $I_A(x) = +\infty$ if $x \in A$ and $I_A(x) = 0$ otherwise.
- ☐ If $\{f_n\colon [0,1] \to [0,\infty)\}_{n\geq 1}$ is a sequence of nonnegative measurable functions which converges to zero almost everywhere, then $\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = 0$.
- ☐ If $\{f_n : \mathbb{R} \to \mathbb{R}\}_n$ is a sequence of nonnegative measurable functions such that $\lim_{n\to\infty} \int_\mathbb{R} f_n(x)dx = 0$, then the pointwise limit of f_n is zero.
- ☐ If $f\colon \mathbb{R} \to \mathbb{R}$ is a nonnegative measurable function, then $\lim_{n\to\infty} \int_\mathbb{R} n\log(1 + \frac{f(x)}{n})dx = \int_\mathbb{R} f(x)dx$.

Question 9 : Let Ω be an open set of $\mathbb{R}$. Assume that $m(\Omega) < +\infty$. Which of the following statements is true ?

- ☐ Define for $x \in \mathbb{R}$, $\varphi(x) = e^{\frac{1}{x^2-1}}\mathbf{1}_{\{y\in\mathbb{R}:|y|<1\}}(x)$. If $f \in C_c^0(\mathbb{R})$, then $f * \varphi \in C_c^\infty(\mathbb{R})$.
- ☐ The Schwartz space $\mathcal{S}(\mathbb{R})$ is not dense in $L^1(\mathbb{R})$.
- ☐ For every $f \in L^2(\Omega)$ and every $\varepsilon > 0$ there is a measurable set $A \subseteq \Omega$ and a constant $c \in \mathbb{R}$ such that $\|f - c\mathbf{1}_A\|_{L^2(\Omega)} \leq \varepsilon$.
- ☐ For every $f \in L^\infty(\Omega)$ there exists a sequence $\{f_n\}_n \subseteq C_c^\infty(\Omega)$ such that $\lim_{n\to+\infty} \|f_n - f\|_{L^\infty(\Omega)} = 0$.

Question 10 : Consider the partial differential equation

$$-\Delta u = 0 \text{ in } \mathbb{R}^2.$$

Which of the following functions $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution of this equation ?

- ☐ $u(x, y) = x^2 + y^2$.
- ☐ $u(x, y) = x^3 - xy^2$.
- ☐ $u(x, y) = ax^3 - \frac{3}{2}ax^2y^2 + 2$ for all $a \in \mathbb{R}$.
- ☐ $u(x, y) = ax^3 - 3axy^2 + x^2 - y^2$ for all $a \in \mathbb{R}$.

Part 2: open questions (35 points)

- Please answer the question in the designated empty space below each exercise.
- If you don't have enough space, you may use the additional empty pages at the end of the exam. In this case, please **mark very clearly** (if possible by indicating the page numbers) i) on the page of the relevant exercise that you are continuing the solution elsewhere and ii) which exercise you are continuing on the additional pages.
- Your answer should be carefully justified. The derivation of the results must be clear and complete.
- Leave the check-boxes empty; they are used for grading only.
- Any solutions not in the booklet will not be graded.

Exercise 1 (*6 points*)

☐0 ☐.5 ☐1 ☐.5 ☐2 ☐.5 ☐3 ☐.5 ☐4 ☐.5 ☐5 ☐.5 ☐6 *Do not write here.*

Let Ω be a measurable set of $\mathbb{R}^d$ be such that $m(\Omega) < +\infty$.

i. (*3 points*) Let $f, g : \Omega \to [0, \infty]$ be two measurable functions such that $fg \geq 1$ almost everywhere in Ω. Show that

a.
$$m(\Omega) \leq \left(\int_\Omega f(x)^4 dx\right)^{1/4} \left(\int_\Omega g(x)^{4/3} dx\right)^{3/4},$$

b.
$$m(\Omega)^2 \leq \int_\Omega f(x)dx \int_\Omega g(x)dx.$$

ii. (*3 points*) Let $f : \Omega \to [0, \infty]$ be such that $\|f\|_{L^\infty(\Omega)} \leq 1$. Show that

$$\lim_{p \to 0^+} \int_\Omega |f(x)|^p dx = m(\{x \in \Omega : f(x) > 0\}).$$

Exercise 2 (*6 points*)

0 .5 1 .5 2 .5 3 .5 4 .5 5 .5 6 *Do not write here.*

Let $f : [0,1] \to \mathbb{R}$ be a measurable and absolutely integrable function.

i. (*1.5 points*) For $n \geq 1$ show that the function $x \mapsto f_n(x) = (xn+1)^2 f(x)$ is measurable and absolutely integrable in $[0,1]$.

ii. (*1.5 points*) For $n \geq 1$ and $x \in [0,1]$, define $g_n(x) = (xn+1)^2|f(x)|$. Show that the sequence $\{g_n\}_n$ converges pointwise in $[0,1]$ and compute its limit.

iii. (*3 points*) Assume in addition that $f \geq 0$ almost everywhere in $[0,1]$ and there exists a constant $C > 0$ such that $\int_{[0,1]}(xn+1)^2 f(x)dx \leq C$ for every $n \geq 1$. Show that $f = 0$ almost everywhere in $[0,1]$.

Exercise 3 (*8 points*)

0 .5 1 .5 2 .5 3 .5 4
.5 5 .5 6 .5 7 .5 8

Do not write here.

Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function such that

$$f(x) = \begin{cases} a(x+\pi) & \text{if } x \in [-\pi, 0) \\ -(x-\pi) & \text{if } x \in [0, \pi). \end{cases}$$

i. *(1 point)* For which value(s) of $a \in \mathbb{R}$ is the function f even ? For each such a value of a, what can you infer on the real Fourier coefficients of f ? Justify.

ii. *(2.5 points)* Compute the Fourier coefficients of f and determine the Fourier series of f.

iii. Answer to the following questions:

   a. *(0.5 point)* Provide all values of $a \in \mathbb{R}$ (if any) for which the Fourier series converges in $L^2((-\pi, \pi))$.

   b. *(1 point)* Provide all values of $a \in \mathbb{R}$ (if any) for which the Fourier series converges pointwise on $\mathbb{R}$.

   c. *(0.5 point)* Provide all values of $a \in \mathbb{R}$ (if any) for which the Fourier series converges uniformly on $\mathbb{R}$.

   In each case, justify your answer and determine the limit of the Fourier series in the most precise and simplified way.

iv. *(2.5 points)* Compute

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

Exercise 4 (*6 points*)

0 .5 1 .5 2 .5 3 .5 4 .5 5 .5 6 *Do not write here.*

i. *(1 point)* State Plancherel's identity (provide all hypotheses needed for the identity to hold).

ii. *(3 points)* Prove Plancherel's identity assuming $\hat{f} \in L^2(\mathbb{R})$, in addition to the hypotheses stated in i. *Hint: you can use, without proof, that for $\varphi \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(x)dx = |\lambda| \int_{\mathbb{R}} \varphi(\lambda x + a)dx$ for all $a \in \mathbb{R}$ and all $\lambda \in \mathbb{R} \setminus \{0\}$.*

iii. *(2 points)* Compute the value of the integral

$$\int_{-\infty}^{\infty} \frac{(t\cos(t) - \sin(t))^2}{t^4} dt.$$

Hint: consider the function $f(x) = x\mathbf{1}_{(-1,1)}(x)$ and compute its Fourier transform.

Exercise 5 *(9 points)*

0 .5 1 .5 2 .5 3 .5 4 .5 5 .5 6 .5 7 .5 8 .5 9 *Do not write here.*

Let $q\colon \mathbb{R} \to \mathbb{R}$ be continuous, and $G\colon \mathbb{R} \to \mathbb{R}$ be in $\mathcal{S}(\mathbb{R})$. Define $Q(t) = \int_0^t q(\tau)d\tau$. Consider the problem

$$\begin{aligned} &\partial_y^2 u(y,t) + \partial_y u(y,t) + q(t)u(y,t) = \partial_t u(y,t), \qquad y \in \mathbb{R}, t > 0, \\ &u(y,0) = G(y). \end{aligned} \tag{1}$$

i. *(3.5 points)* Use the Fourier transform to show formally that

$$u(y,t) = \int_{-\infty}^{\infty} e^{R(\xi,t)+2\pi i \xi y} \hat{G}(\xi) d\xi \tag{2}$$

where $R(\xi,t) = (2\pi i\xi)^2 t + t(2\pi i \xi) + Q(t)$.

ii. *(1 point)* Show that the integrand $\xi \mapsto e^{R(\xi,t)+2\pi i\xi y}\hat{G}(\xi)$ is absolutely integrable for any fixed $t > 0$ (and as a result the solution (2) is well-defined).

Additionally, show that $u(y,t) = (K_t * G)(y)$ where, for fixed $t > 0$, the kernel $K_t\colon \mathbb{R} \to \mathbb{R}$ is given by

$$K_t(y) = \int_{-\infty}^{\infty} e^{R(\xi,t)+2\pi i \xi y} d\xi.$$

iii. *(2 points)* Let $G(y) = \frac{1}{2\sqrt{\pi}} e^{-y^2/4}$ for all $y \in \mathbb{R}$. Show that

$$u(y,t) = \frac{1}{2\sqrt{\pi}\sqrt{t+1}} e^{Q(t) - \frac{(t+y)^2}{4(t+1)}}.$$

Hint: You can use without proof that for any $a > 0, b \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} e^{-(a\xi - b)^2} d\xi = \frac{\sqrt{\pi}}{a}.$$

iv. *(2.5 points)* Fix $T > 0$, and consider the following PDE, known as the Black-Scholes equation:

$$\begin{aligned} &\partial_t v(s,t) + s\partial_s v(s,t) + \frac{s^2}{2}\partial_s^2 v(s,t) - v(s,t) = 0, \qquad s \geq 0, t \in [0,T], \\ &v(0,t) = 0, \qquad v(s,T) = f(s), \end{aligned}$$

where we define

$$f(s) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}\log^2(s)}, \qquad s > 0.$$

Formally compute an explicit expression for $v(s,t)$.
Hint: Consider a change of variables $(s,t) \mapsto (y,\tau)$ of the form

$$s = e^{ay}, \qquad t = T - b\tau$$

for an appropriate choice of parameters $a, b > 0$, and then use part iii.

+98/18/55+

+98/22/51+

+98/23/50+

Table of Fourier transform pairs

	$f(y)$	$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1, & \text{if } \lvert y\rvert < \lvert b\rvert \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\alpha} - e^{-ic\alpha}}{i\alpha}$
3	$f(y) = \begin{cases} e^{-wy}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases} \quad (w > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{w + i\alpha}$
4	$f(y) = \begin{cases} e^{-wy}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(w+i\alpha)b} - e^{-(w+i\alpha)c}}{w + i\alpha}$
5	$f(y) = \begin{cases} e^{-iwy}, & \text{if } b < y < c \\ 0, & \text{otherwise} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(w+\alpha)b} - e^{-i(w+\alpha)c}}{w + \alpha}$
6	$f(y) = \frac{1}{y^2 + w^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert w\alpha\rvert}}{\lvert w\rvert}$
7	$f(y) = \frac{e^{-\lvert wy\rvert}}{\lvert w\rvert} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha^2 + w^2}$
8	$f(y) = e^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\lvert w\rvert} e^{-\frac{\alpha^2}{4w^2}}$
9	$f(y) = ye^{-w^2y^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \frac{-i\alpha}{2\sqrt{2}\lvert w\rvert^3} e^{-\frac{\alpha^2}{4w^2}}$
10	$f(y) = \frac{4y^2}{(y^2 + w^2)^2} \quad (w \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert w\rvert} - \lvert\alpha\rvert\right) e^{-\lvert w\alpha\rvert}$