Teacher: Maria Colombo
Analysis IV - Exam - MA
18/06/2025
180 minutes

X-5

Student 5

SCIPER : **XXXXX** Room : **PO 01** Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 20 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A4 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  +1 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

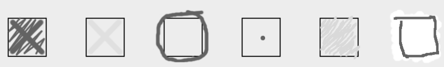
Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte



First part: multiple choice questions

For each question, mark the box corresponding to the correct answer.

Question 1 Let $f \in C^1(\mathbb{R})$ be such that $f(0) = 2,\ f'(0) = 1$. The limit $\lim_{n\to\infty} \int_0^1 f(x/n)dx$ is equal to

- ☐ 2.
- ☐ 0.
- ☐ It depends on the function.
- ☐ 1.

Question 2 Let $f(x) = e^{-x^2}x^{-1/3}$. Then, $f \in L^p((0, +\infty))$ if and only if

- ☐ $p \in [1, 3)$.
- ☐ $p \in [1, 2)$.
- ☐ $p = \infty$.
- ☐ $p \in [1, \infty)$.

Question 3 Let

$$K = \left\{ f \in C^1(\mathbb{R}, \mathbb{C}) : f \text{ is 1-periodic}, \int_0^1 f(x)dx = 0, \int_0^1 |f|^2(x)dx = 1 \right\}.$$

What is the value of

$$\inf \left\{ \left(\int_0^1 |f'|^2(x)dx \right)^{1/2} : f \in K \right\} \quad ?$$

- ☐ 0.
- ☐ 2π.
- ☐ π.
- ☐ 1.

Question 4 Let $f(x) = \frac{x}{1+x^4}$ and let $\hat{f}$ be its Fourier Transform. Which of the following statements is **true**?

- ☐ $Im(\hat{f}) = 0$ and $\hat{f}$ is odd.
- ☐ $Re(\hat{f}) = 0$ and $\hat{f}$ is even.
- ☐ $Im(\hat{f}) = 0$ and $\hat{f}$ is even.
- ☐ $Re(\hat{f}) = 0$ and $\hat{f}$ is odd.

Question 5 Let $\Omega := \{(x_1, x_2, x_3) : |x_1| \le |x_2| \le |x_3| \le 1\} \subset \mathbb{R}^3$. Then, $m(\Omega)$ is equal to

- ☐ 1/27.
- ☐ 1/6.
- ☐ 4/3.
- ☐ 1.

Question 6 $\lim_{n\to\infty}\int_0^1 \frac{n}{1+nx^3}dx$ is equal to

- ☐ 1.
- ☐ 1/2.
- ☐ ∞.
- ☐ 0.

Question 7 Let $E=[0,3]\cup\bigcup_{n\in\mathbb{N}}[n,n+2^{-n}]$. Then $m(E)$ is equal to

- ☐ 25/8.
- ☐ ∞.
- ☐ 13/4.
- ☐ 5.

Question 8 The function $v(x,y)=x^3y+axy^3$ solves $\Delta v(x,y)=0$ if and only if

- ☐ $a=0$.
- ☐ $a=-1$.
- ☐ $a=2$.
- ☐ $a=1$.

Question 9 Let $E\subset[0,1]$ be the set of numbers that, written in ternary expansion, have 1 as first digit and 2 as third digit, that is $E:=\{0.1a_2 2a_4a_5...\}$. What is the measure of E?

- ☐ 0.
- ☐ 1/81.
- ☐ 1/9.
- ☐ 10/81.

Question 10 Let f be the $2\pi-$ periodic function $f(x)=\sin(3x/2)$ on $[-\pi,\pi]$, and let a_n and b_n be its real Fourier coefficients, namely $a_n=\pi^{-1}\int_{-\pi}^{\pi}f(x)\cos(nx)dx$ and $b_n=\pi^{-1}\int_{-\pi}^{\pi}f(x)\sin(nx)dx$. Which of the following statements is **true**?

- ☐ $\sum_{n\geq1}(-1)^n b_n=0$.
- ☐ the Fourier series converges uniformly.
- ☐ $\sum_{n\geq1}|b_n|^2=4\pi^2$.
- ☐ $a_n,b_n\leq C/n^3$ for some $C>0$, $\forall n\in\mathbb{N}$.

Second part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 11: *This question is worth 6 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

Let $f : \mathbb{R} \to (0, \infty)$ be the function

$$f(t) := \int_0^\infty e^{-x^2 - \frac{t^2}{x^2}} \, dx.$$

(a) (**2 points**). Prove that f is continuous, i.e., for every sequence $\{t_n\} \subset \mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ we have $f(t_n) \to f(t)$.

(b) (**2 points**). Prove that for every $t > 0$, f is differentiable in t and

$$f'(t) = -2t \int_0^\infty \frac{1}{x^2} e^{-x^2 - \frac{t^2}{x^2}} \, dx.$$

(c) (**2 points**). Using the change of variables $y = t/x$ prove that $\lim_{t \downarrow 0} f'(t) = -\sqrt{\pi}$.

Question 12: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and let $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ be its Fourier coefficients.

(a) (**4 points**). State and prove the theorem about the L^2 convergence of the Fourier Series (you can use Weierstrass Theorem without proof).

(b) (**2 points**). Prove that f is $(1/2)$-periodic if and only if $\hat{f}(n) = 0$ for every n odd.

Question 13: *This question is worth 4 points.*

☐0 ☐1 ☐2 ☐3 ☐4

Let $f \in L^1((0,1))$ with $f \geq 0$ almost everywhere. For every $n \in \mathbb{N}$ define

$$E_n := \{x \in (0,1) : n \leq f(x) < n+1\}.$$

(a) (**2 points**). Compute $\lim_{n\to\infty} nm(E_n)$.

(b) (**2 points**). Prove the following inequalities:

$$\int_{(0,1)} f(x)dx \leq \sum_{n=0}^{\infty}(n+1)m(E_n) \leq \int_{(0,1)} f(x)dx + 1.$$

Question 14: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

(a) (**1 point**). Find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1((0,1))$ such that $\|f_n\|_{L^1} \leq 1$, $f_n(x) \to 0$ a.e. and $\int_0^1 f_n = 1$.

(b) Let $p \in [1,\infty]$ and $u_n : (0,1) \to \mathbb{R}$ be a sequence of measurable functions such that

$$\|u_n\|_{L^p} \leq 1 \quad \text{for every } n \in \mathbb{N}, \qquad u_n(x) \to 0 \quad \text{for a.e. } x.$$

(a) (**1 point**). Prove that if $p = \infty$, then $\int_0^1 u_n \to 0$.

(b) (**3 points**). Prove that if $p > 1$, then $\int_0^1 u_n \to 0$.
Hint for part (b) of (b): for any $M > 0$, write $u_n = u_n \mathbb{1}_{\{u_n \leq M\}} + u_n \mathbb{1}_{\{u_n > M\}}$ and argue on each addend separately.

Question 15: *This question is worth 4 points.*

☐0 ☐1 ☐2 ☐3 ☐4

By means of the Fourier Transform, find all the functions $u \in L^1(\mathbb{R})$ such that $u * u = \frac{1}{\sqrt{18\pi}} e^{-\frac{x^2}{18}}$.

Question 16: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8

Let $g : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function such that $g(x) = \frac{1}{12}x(x-\pi)(x-2\pi)$ in $[0, 2\pi)$, and let

$$\sum_{n=0}^{\infty} \bar{a}_n \cos(nx) + \sum_{n=1}^{\infty} \bar{b}_n \sin(nx)$$

be its Fourier series.

(a) (**1 point**). Does the Fourier series of g converge to g in L^2?

(b) (**1 point**). Prove that $\bar{a}_n = 0$ for every $n \geq 0$.

(c) (**2 points**). Compute the coefficient $\bar{b}_n$ for every $n \geq 1$.

(d) (**2 points**). Consider the following problem on the spatial interval $[0, \pi]$:

$$\begin{cases} u_{tt} + 2u_t = u_{xx} & \text{for } (t,x) \in (0,\infty) \times [0,\pi], \\ u(t,0) = u(t,\pi) & \text{for } t \in (0,\infty), \\ u(0,x) = 0 & \text{for } x \in [0,\pi], \\ u_t(0,x) = g(x) & \text{for } x \in [0,\pi]. \end{cases} \tag{1}$$

Find functions $b_n : [0,\infty) \to \mathbb{R}$ such that

$$u(t,x) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$$

is a formal solution of (1).

Hint: Search the general solution of the ode $y'' + 2y' + ky = 0$ in the form $y(t) = e^{-t}(c_1 + c_2 t)$ or $y(t) = e^{-t}\left(c_1 \sin(\omega t) + c_2 \cos(\omega t)\right)$, for c_1, c_2 to be suitably chosen.

(e) (**2 points**). Prove that the solution found in point (d) is $C^1\left([0,\infty) \times [0,\pi]\right)$.

+124/18/43+