EPFL

Teacher : Maria Colombo
Analysis IV (MATH-205) - XX
19/06/2024
Duration : 180 minutes

87

Student 1

SCIPER : **100000**

Do not turn the page before the start of the exam. This document is double-sided, has 20 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A4 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  - +1 points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - 0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Which of the following statements is **true**?

- ☐ The set of points where the function $\mathbb{1}_{[0,+\infty)}$ is not continous has measure zero.
- ☐ The function $\mathbb{1}_{[0,+\infty)}$ is a.e. equal to a continuous function.
- ☐ There exists a simple function that coincides a.e. with a continuous non-constant function.
- ☐ If $f : \mathbb{R} \to \mathbb{R}$ is a.e. equal to a continuous function, then f is a.e. continuous.

Question 2 $f(x) = \log(x) \in L^p((0, 1))$ if and only if

- ☐ $p \in [1, +\infty)$.
- ☐ $p \in (1, +\infty]$.
- ☐ $p \in (1, 2]$.
- ☐ $p = 1$.

Question 3 Let A be the set of those numbers in $(0, 1)$ that have a decimal expansion containing at least one digit 7. Which of the following statements is **true**?

- ☐ A is not measurable.
- ☐ A is measurable and $0 < m(A) < 1$.
- ☐ A is measurable and $m(A) = 0$.
- ☐ A is measurable and $m(A) = 1$.

Question 4 Consider the Fourier series $\sum_{n=1}^{+\infty} \frac{1}{n^3} \cos(nx)$. Which of the following statements is **true**?

- ☐ It is the Fourier series of a function $u \in C^3(\mathbb{R})$, but not in $C^\infty(\mathbb{R})$.
- ☐ It is the Fourier series of a function $u \in C^1(\mathbb{R})$, but not in $C^3(\mathbb{R})$.
- ☐ It converges pointwise, but not uniformly, on $\mathbb{R}$.
- ☐ It cannot be the Fourier series of any $2\pi-$ periodic function in $L^2(0, 2\pi)$.

Question 5 Let C denote the Cantor set. Which of the following statements is **true**?

- ☐ If $E \subseteq \mathbb{R}$ is a measurable set, then $m(E \cap [x, \infty)) \to 0$ as $x \to +\infty$.
- ☐ For any $E \subseteq \mathbb{R}$, it holds $m^*(E) = m^*(\bar{E})$.
- ☐ If $E \subseteq \mathbb{R}$ is not measurable, then $E \cap C$ can be not measurable.
- ☐ If $E \subseteq \mathbb{R}$ is not measurable, then $E \cap C^c$ can be not measurable.

Question 6 Let f_1 be a smooth, compactly supported function, $f_2(x) = e^{-x^2}$, $f_3(x) = \mathbb{1}_{[-1,1]}(x)$ for $x \in \mathbb{R}$. The Fourier transform belongs to L^1:

- ☐ for one of the three functions.
- ☐ for none of the three functions.
- ☐ for all of the three functions.
- ☐ for two of the three functions.

Question 7 Let $u : \mathbb{R}^2 \to \mathbb{R}$ solving $\Delta u = xy$ and $v(x, y) = \frac{1}{2}u(2x, 2y) + 2x^2 - y^2$. Then, v solves:

- ☐ $\Delta v = 8xy - 2$.
- ☐ $\Delta v = 2xy + 2$.
- ☐ $\Delta v = 2xy - 2$.
- ☐ $\Delta v = 8xy + 2$.

Question 8 What are the real Fourier coefficients $\{a_n\}$ and $\{b_n\}$ of $\operatorname{sign}(\sin(x))$?

- ☐ $a_n = 0$, $b_{2n} = \frac{4}{n\pi}$, $b_{2n+1} = 0$.
- ☐ $a_{2n} = \frac{4}{n\pi}$, $a_{2n+1} = 0$, $b_{2n} = 0$, $b_{2n+1} = \frac{4}{(2n+1)\pi}$.
- ☐ $a_n = 0$, $b_{2n} = 0$, $b_{2n+1} = \frac{4}{(2n+1)\pi}$.
- ☐ $a_{2n} = 0$, $a_{2n+1} = \frac{4}{n\pi}$, $b_n = 0$.

Question 9 Let $F(x) = \left|\sum_{n=0}^{4} e^{2\pi i n x}\right|^2$. Which of the following statements is **true**?

- ☐ $\int_0^1 F(x)dx = 5$.
- ☐ $\forall \varepsilon > 0 \quad \exists \delta < 1/2$ such that $|F(x)| \le \varepsilon$ for all $x \in [\delta, 1 - \delta]$.
- ☐ F is not a trigonometric polynomial.
- ☐ $\int_0^1 F(x)^2 dx = 25$.

Question 10 Let f, g be two $1-$periodic functions in $L^2([0, 1])$. Assume that $f * g = 0$. Then:

- ☐ f, g are orthogonal in $L^2([0, 1])$, namely $\int_0^1 fg = 0$.
- ☐ Either $f = 0$ or $g = 0$.
- ☐ f and g have the same L^2- norm, namely $\|f\|_{L^2} = \|g\|_{L^2}$.
- ☐ $fg = 0$.

Second part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 11: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

(a) (3 points). State and prove Fatou's lemma.

(b) (1 point). Give an example of a sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n : [0,1] \to \mathbb{R}$, satisfying the assumptions of Fatou's lemma and such that the *strict inequality* occurs.

(c) (2 points). Let f be a non-decreasing function on $[0,1]$ and assume that for a.e. $x \in [0,1)$ there exists the following limit:

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}.$$

Prove that

$$\int_0^1 f'(x)\,\mathrm{d}x \leqslant f(1) - f(0).$$

Question 12: *This question is worth 4 points.*

☐0 ☐1 ☐2 ☐3 ☐4

Let $\{f_n\}_{n=1}^{+\infty}$ be a sequence of functions defined as follows:

$$f_n(x) = \begin{cases} \frac{n\sqrt{x}}{1+n^2x^2}, & x \in [0,1), \\ e^{-x^2/n}, & x \in [1,+\infty). \end{cases}$$

Compute $\lim_{n\to+\infty} f_n(x)$ for all $x \in [0,+\infty)$,

$$\lim_{n\to+\infty} \int_1^{+\infty} f_n(x)\,\mathrm{d}x \quad \text{and} \quad \lim_{n\to+\infty} \int_0^1 f_n(x)\,\mathrm{d}x.$$

Question 13: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $f : (0,1) \to \mathbb{R}$ be a measurable function and let $1 \leq p < +\infty$.

(a) (1 point). Prove that, for all $y > 0$, we have

$$y^p m(\{x \in (0,1) : |f(x)| \geqslant y\}) \leqslant \int_0^1 |f(x)|^p \,\mathrm{d}x\,.$$

(b) (2 points). Show that

$$\int_0^1 |f(x)|^p \,\mathrm{d}x = p \int_0^\infty y^{p-1} m(\{x \in (0,1) : |f(x)| \geqslant y\}) \,\mathrm{d}y\,.$$

Hint: Notice that $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} \,\mathrm{d}y$.

(c) (2 points). Let $1 \leq q < p < +\infty$ and assume that f satisfies

$$y^p m(\{x \in (0,1) : |f(x)| \geqslant y\}) \leqslant 1, \quad \text{for all } y > 0. \tag{1}$$

Show that $f \in L^q((0,1))$.

(d) (1 point). Consider $f(x) = x^{-1/p}$, for $1 \leq p < +\infty$. Show that f satisfies (1), but $f \notin L^p((0,1))$.

Question 14: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Consider the $2\pi-$periodic function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \frac{e^x + e^{-x}}{2} \qquad \text{for } x \in (-\pi, \pi],$$

(a) (1 point). Compute $\|f\|_{L^2((-\pi,\pi])}$.

(b) (1 point). Determine for which values of $x \in (-\pi, \pi]$ the Fourier Series of f converges pointwise to $f(x)$.

(c) (2 points). Compute the Fourier series of f.

(d) (1 point). Compute the value of

$$\sum_{k=0}^{+\infty} \frac{1}{(1+k^2)^2}.$$

Question 15: *This question is worth 5 points.*

□0 □1 □2 □3 □4 □5

Let f be a 1-periodic function on $\mathbb{R}$ such that f belongs to $L^2((0,1))$. Let $\{c_n\}_{n\in\mathbb{Z}}$ be the complex Fourier coefficients of f. For every $h \in \mathbb{R}$, let us define the function f_h by $f_h(x) = f(x-h)$.

(a) (1 point). Give the Fourier series of f_h in terms of $\{c_n\}_{n\in\mathbb{Z}}$ and h.

(b) (1 point). Find all 1−periodic C^1 functions that satisfy $f'(x) = f(x-1/2)$ for all $x \in \mathbb{R}$.

(c) (3 points). Prove that

$$\liminf_{h\to 0} \frac{\|f_h - f\|_{L^2((0,1))}}{|h|} > 0,$$

unless f is constant almost everywhere.

Question 16: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8

Let $g \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$. Let us consider the following Cauchy problems:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x u^\varepsilon(t,x) - \varepsilon \partial^2_{xx} u^\varepsilon(t,x) = 0, & t > 0,\ x \in \mathbb{R}, \\ u^\varepsilon(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{CP$_\varepsilon$}$$

and

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{CP}$$

(a) (2 point). For a fixed $\varepsilon > 0$, write the formal solution u^ε of (CP$_\varepsilon$).

(b) (1 point). Use the Fourier transform to write a formula for the solution u of (CP), showing that $\forall t > 0$ $u(t,\cdot)$ is given by a suitable translation of g.

(c) (2 points). Prove that, for $\varepsilon > 0$,

$$\|u^\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le \|g\|_{L^2(\mathbb{R})} \quad \text{for all } t > 0.$$

(d) (1 point). Show that $u^\varepsilon \in C^\infty((0,+\infty) \times \mathbb{R})$.

(e) (2 points). Prove that the solutions u^ε of (CP$_\varepsilon$) converge to the solution u of (CP) in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, namely, for every $t > 0$,

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } \varepsilon \to 0.$$