Teacher : Maria Colombo
Analysis 4 (MATH-205)
21/06/2023
Duration : 180 minutes

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Student 1

SCIPER : **999000**

Do not turn the page before the start of the exam. This document is double-sided, has 20 pages, the last ones possibly blank. Do not unstaple.

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- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A4 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  +1 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
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choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 Let $f \in C(\mathbb{R}/\mathbb{Z})$ be a 1-periodic function and let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ be its real Fourier coefficents, namely $a_n = 2\int_0^1 f(x)\cos(2\pi nx)dx$, $b_n = 2\int_0^1 f(x)\sin(2\pi nx)dx$.
Assume that $f(x+1/2) = -f(x)$ for all x. Which of the following is **true**?

- ☐ $a_{2m+1} = b_{2m+1} = 0$ for all $m = 1, 2, \ldots$.
- ☐ $a_n = b_n = 0$ for all $n = 1, 2, \ldots$.
- ☐ $a_0 = 0$ and $a_{2m} = b_{2m} = 0$ for all $m = 1, 2, \ldots$.
- ☐ None of the others is true.

Question 2 Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $\hat{f}(\xi) = |\xi|^{5/2}\mathrm{e}^{-|\xi|^2/4}, \hat{g}(\xi) = \mathrm{e}^{-|\xi|^2/4}$. Which of the following is **true**?

- ☐ $g \notin \mathcal{S}(\mathbb{R})$.
- ☐ $f * g \notin L^\infty(\mathbb{R})$
- ☐ $f \notin \mathcal{S}(\mathbb{R})$.
- ☐ f and g have compact support.

Question 3 Let us consider the following PDE

$$\partial_x u - u\partial_y u = 0\,, \tag{1}$$

for $u : \mathbb{R}^2 \to \mathbb{R}$. Assume that v, w solve (1). Which of the following statement is **true**?

- ☐ For any v, w solutions of (1), $v - w$ solves (1).
- ☐ For any v, w solutions of (1), $v + w$ solves (1).
- ☐ For any v solution of (1) and any $\lambda > 0$, $v_\lambda(x, y) = \lambda v(\lambda x, y)$ solves (1).
- ☐ $v(x, y) = x + xy + y^3 + x^2 y$ is a solution of (1).

Question 4 Let $f(x) = x^2 e^{-x^2}$, and denote with $\hat{f}(\xi)$ its Fourier Transform. Which of the following statements is **true**?

- ☐ $\lim_{\xi\to\pm\infty} \hat{f}(\xi) = 1$
- ☐ $\hat{f}(\xi) \in L^1 \cap L^\infty$
- ☐ $\hat{f}(\xi) \in L^2(\mathbb{R})$ but $\hat{f}(\xi) \notin L^1(\mathbb{R})$
- ☐ $\hat{f}(\xi)$ has compact support in $\mathbb{R}$

Question 5 For every $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) = \dfrac{1}{x^2}\mathbb{1}_{[2^{-n}, 2^{-n+1}]}$.
Which of the following statements is **true**?

- ☐ $\limsup_n \int_{\mathbb{R}} f_n\, dx > \int_{\mathbb{R}} \limsup_n f_n\, dx$.
- ☐ $\lim_n \int_{\mathbb{R}} |f_n - \lim_n f_n|\, dx = 0$.
- ☐ $\lim_n \int_{\mathbb{R}} f_n\, dx < \int_{\mathbb{R}} \liminf_n f_n\, dx$.
- ☐ $\lim_n \int_{\mathbb{R}} f_n\, dx = \int_{\mathbb{R}} \lim_n f_n\, dx$.

Question 6 Consider $f_n(x) = x^{-1/3}\mathbb{1}_{[0,1/n]}$. Then $f_n \to 0$

- [ ] in L^1 but not in L^2.
- [ ] in L^4 but not in L^∞.
- [ ] in L^2 but not in L^4.
- [ ] in L^∞.

Question 7 Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \geq 0$. Which of the following statements is **true**?

- [ ] If $f \in L^3(\mathbb{R})$ then $f^{3/2} \in L^2(\mathbb{R})$.
- [ ] If $f \in L^2(\mathbb{R})$ then $f \in L^1(\mathbb{R})$.
- [ ] If $f, g \in L^1(\mathbb{R})$ then $fg \in L^1(\mathbb{R})$.
- [ ] If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$.

Question 8 Let $f, g \in \mathcal{S}(\mathbb{R})$ and $\hat{f}, \hat{g}$ be their Fourier transform. Which of the following inequality is **true**?

- [ ] $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^\infty}\|\hat{g}\|_{L^1}$
- [ ] $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^2}\|\hat{g}\|_{L^2}$
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Question 9 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : (0,1) \to \mathbb{R}$. Which of the following is **true**?

- [ ] If $f_n(x) \leq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then
$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$
- [ ] If $f_n(x) \geq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then
$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$
- [ ] Let $M \in \mathbb{N}$, if $f_n(x) \geq 0$ for any $x \in (0,1)$ and for any $n \leq M$, then
$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$
- [ ] Let $c \in \mathbb{R}$, if $f_n(x) \leq c$ for any $x \in (0,1)$ and for any $n \in \mathbb{N}$ then
$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$

Question 10 Which of the following sets has infinite outer measure?

- [ ] $\bigcup_{n=1}^{\infty} \left(n-3, \frac{(n-1)^3}{n^2}\right)$
- [ ] $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$
- [ ] $\bigcup_{n=1}^{\infty} \left(-\cos^2(\frac{1}{n}), \sin^2(\frac{1}{n})\right)$
- [ ] $\bigcup_{n=1}^{\infty} \left(2^n, \frac{2^{2n}+1}{2^n}\right)$

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 11: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Let $A = (1, \infty) \times [0, \pi/2]$, $n \in \mathbb{N}$ and $f_n : A \to \mathbb{R}$ defined as

$$f_n(x, y) = \frac{x + \cos(2y)}{(x^2 + 1 + \frac{y}{n})^2}.$$

Compute $\lim_{n\to\infty} f_n(x, y)$ and show that

$$\lim_{n\to\infty} \int_A f_n(x, y) dx\, dy = \frac{\pi}{8}.$$

Question 12: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Let $f \in L^1(\mathbb{R})$. Then, prove the following:

(a) *(2 points).*

$$\lim_{n\to\infty} \int_n^\infty |f(x)|dx = 0\,.$$

(b) *(1 point).*

$$\liminf_{x\to\infty} |f(x)| = 0\,.$$

(c) *(2 points).* There exists a function $f \in L^1(\mathbb{R})$ such that

$$\limsup_{x\to\infty} |f(x)| = \infty\,.$$

Question 13: *This question is worth 5 points.*

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Consider two non-negative simple functions $f, g : [0,1] \to \mathbb{R}$, namely $f(x) = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}(x)$, $g(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{C_i}(x)$ where $N \in \mathbb{N}$ and $a_i \geq 0\,, c_i \geq 0$, $C_i, A_i \subset [0,1]$ are measurable sets for any $i = 1, \ldots, N$. We associate to f, g the two functions

$$F(x) = \sum_{i=1}^{N} a_i \mathbb{1}_{[0,m(A_i)]}(x), \qquad G(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{[0,m(C_i)]}(x),$$

Prove the following:

(a) *(3 points).*

$$\int_0^1 |f(x)|dx = \int_0^1 |F(x)|dx$$
$$\int_0^1 f(x)g(x)dx \leq \int_0^1 F(x)G(x)dx$$

(b) *(2 points).* Finally, for $N = M = 1$ and $A_1 \cap C_1 = \emptyset$ prove that

$$\int_0^1 |F(x) - G(x)|dx \leq \int_0^1 |f(x) - g(x)|dx.$$

Question 14: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $\sum_{n=-\infty}^{\infty}|a_n| < \infty$ and define $\varphi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$. Consider the PDE

$$\begin{cases} \partial_t u = 2\partial_{xx}u + u - \varphi & (t,x) \in (0,\infty)\times(-\pi,\pi)\,, \\ u(0,x) \equiv 0 & \text{for all } x \in [-\pi,\pi) \\ u(t,-\pi) = u(t,\pi) & \text{for all } t > 0\,, \end{cases} \tag{2}$$

(a) *(1 point).* Prove that the sequence $\{\varphi_N\}_{N\in\mathbb{N}}$ defined as $\varphi_N = \sum_{n=-N}^{N} a_n e^{inx}$ is a Cauchy sequence in $C^0[-\pi,\pi]$ and deduce that $\varphi \in C^0[-\pi,\pi]$.

(b) *(3 points).* Write $u(t,x) = \sum_{n=-\infty}^{\infty} b_n(t)e^{inx}$ and find from (2) an ODE satisfied formally for $b_n(t)$ in terms of a_n. Finally, compute explicitly $b_n(t)$.

(c) *(4 points).* Prove that the solution u found in the previous point is such that $u, \partial_t u, \partial_{xx}u \in C^0((0,\infty)\times[-\pi,\pi))$ and u solves (2) and that $u(t,x) \to 0$ as $t \to 0^+$ uniformly in x.

Hint: Recall that the solution of $\dot{y} = cy + d$ with $c, d \in \mathbb{R}$ and $y(0) = 0$ is $y(t) = \frac{d}{c}[e^{ct} - 1]$.

Question 15: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function such that

$$f(x) = \max\{0, \cos(x)\}$$

Denote with $\{a_n\}$ and $\{b_n\}$ the real Fourier coefficients of f, i.e.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

(a) (*5 points*). Compute the Fourier series of f

(b) (*1 point*). Compute the value of the series

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2-1},$$

Hint: $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$.

Question 16: *This question is worth 6 points.*

$\square_0$ $\square_1$ $\square_2$ $\square_3$ $\square_4$ $\square_5$ $\square_6$

(a) (*2 points*). Let $g \in C_c^\infty(\mathbb{R})$. Give a formula of $\widehat{g^{(k)}}(\xi)$ in terms of $\hat{g}(\xi)$ and prove it for $k = 1$, where $g^{(k)}$ denote the k-th derivative.

(b) (*4 points*). Is there a non-zero function $f \in \mathcal{S}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \xi^k f(\xi)\, d\xi = 0 \qquad \text{for every } k \in \mathbb{N}\,?$$

Hint: for (b) consider $g \in \mathcal{S}(\mathbb{R})$ such that $f = \widehat{g}$.

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2

Student 2

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- ☐ None of the others is true.
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- ☐ $a_{2m+1} = b_{2m+1} = 0$ for all $m = 1, 2, \dots$.
- ☐ $a_n = b_n = 0$ for all $n = 1, 2, \dots$.

Question 2 Let $f(x) = x^2 e^{-x^2}$, and denote with $\hat{f}(\xi)$ its Fourier Transform. Which of the following statements is **true**?

- ☐ $\hat{f}(\xi) \in L^2(\mathbb{R})$ but $\hat{f}(\xi) \notin L^1(\mathbb{R})$
- ☐ $\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 1$
- ☐ $\hat{f}(\xi)$ has compact support in $\mathbb{R}$
- ☐ $\hat{f}(\xi) \in L^1 \cap L^\infty$

Question 3 Consider $f_n(x) = x^{-1/3}\mathbb{1}_{[0,1/n]}$. Then $f_n \to 0$

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Question 5 Let us consider the following PDE

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- ☐ For any v solution of (1) and any $\lambda > 0$, $v_\lambda(x, y) = \lambda v(\lambda x, y)$ solves (1).
- ☐ For any v, w solutions of (1), $v - w$ solves (1).
- ☐ $v(x, y) = x + xy + y^3 + x^2y$ is a solution of (1).

Question 6 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : (0,1) \to \mathbb{R}$. Which of the following is **true**?

- [ ] Let $c \in \mathbb{R}$, if $f_n(x) \leq c$ for any $x \in (0,1)$ and for any $n \in \mathbb{N}$ then

$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$

- [ ] Let $M \in \mathbb{N}$, if $f_n(x) \geq 0$ for any $x \in (0,1)$ and for any $n \leq M$, then

$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$

- [ ] If $f_n(x) \geq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then

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Question 7 Let $f, g \in \mathcal{S}(\mathbb{R})$ and $\hat{f}, \hat{g}$ be their Fourier transform. Which of the following inequality is **true**?

- [ ] $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^\infty}\|\hat{g}\|_{L^2}$
- [ ] $\|f * g\|_{L^2} \leq \|\hat{f}\|_{L^2}\|\hat{g}\|_{L^2}$
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Question 8 For every $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) = \dfrac{1}{x^2}\mathbb{1}_{[2^{-n},2^{-n+1}]}$. Which of the following statements is **true**?

- [ ] $\lim_n \int_\mathbb{R} f_n \, dx = \int_\mathbb{R} \lim_n f_n \, dx$.
- [ ] $\lim_n \int_\mathbb{R} |f_n - \lim_n f_n| \, dx = 0$.
- [ ] $\lim_n \int_\mathbb{R} f_n \, dx < \int_\mathbb{R} \liminf_n f_n \, dx$.
- [ ] $\limsup_n \int_\mathbb{R} f_n \, dx > \int_\mathbb{R} \limsup_n f_n \, dx$.

Question 9 Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $\hat{f}(\xi) = |\xi|^{5/2}\mathrm{e}^{-|\xi|^2/4}$, $\hat{g}(\xi) = \mathrm{e}^{-|\xi|^2/4}$. Which of the following is **true**?

- [ ] $f \notin \mathcal{S}(\mathbb{R})$.
- [ ] f and g have compact support.
- [ ] $f * g \notin L^\infty(\mathbb{R})$
- [ ] $g \notin \mathcal{S}(\mathbb{R})$.

Question 10 Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \geq 0$. Which of the following statements is **true**?

- [ ] If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$.
- [ ] If $f \in L^2(\mathbb{R})$ then $f \in L^1(\mathbb{R})$.
- [ ] If $f \in L^3(\mathbb{R})$ then $f^{3/2} \in L^2(\mathbb{R})$.
- [ ] If $f, g \in L^1(\mathbb{R})$ then $fg \in L^1(\mathbb{R})$.

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 11: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Let $A = (1, \infty) \times [0, \pi/2]$, $n \in \mathbb{N}$ and $f_n : A \to \mathbb{R}$ defined as

$$f_n(x, y) = \frac{x + \cos(2y)}{(x^2 + 1 + \frac{y}{n})^2}.$$

Compute $\lim_{n\to\infty} f_n(x, y)$ and show that

$$\lim_{n\to\infty} \int_A f_n(x, y) dx\, dy = \frac{\pi}{8}.$$

Question 12: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Let $f \in L^1(\mathbb{R})$. Then, prove the following:

(a) *(2 points).*

$$\lim_{n\to\infty} \int_n^\infty |f(x)|dx = 0\,.$$

(b) *(1 point).*

$$\liminf_{x\to\infty} |f(x)| = 0\,.$$

(c) *(2 points).* There exists a function $f \in L^1(\mathbb{R})$ such that

$$\limsup_{x\to\infty} |f(x)| = \infty\,.$$

Question 13: *This question is worth 5 points.*

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Consider two non-negative simple functions $f, g : [0,1] \to \mathbb{R}$, namely $f(x) = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}(x)$, $g(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{C_i}(x)$ where $N \in \mathbb{N}$ and $a_i \geq 0$, $c_i \geq 0$, $C_i, A_i \subset [0,1]$ are measurable sets for any $i = 1, \ldots, N$. We associate to f, g the two functions

$$F(x) = \sum_{i=1}^{N} a_i \mathbb{1}_{[0,m(A_i)]}(x), \qquad G(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{[0,m(C_i)]}(x),$$

Prove the following:

(a) *(3 points).*

$$\int_0^1 |f(x)|dx = \int_0^1 |F(x)|dx$$
$$\int_0^1 f(x)g(x)dx \leq \int_0^1 F(x)G(x)dx$$

(b) *(2 points).* Finally, for $N = M = 1$ and $A_1 \cap C_1 = \emptyset$ prove that

$$\int_0^1 |F(x) - G(x)|dx \leq \int_0^1 |f(x) - g(x)|dx.$$

Question 14: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ and define $\varphi(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$. Consider the PDE

$$\begin{cases} \partial_t u = 2\partial_{xx} u + u - \varphi & (t,x) \in (0,\infty) \times (-\pi,\pi)\,, \\ u(0,x) \equiv 0 & \text{for all } x \in [-\pi,\pi) \\ u(t,-\pi) = u(t,\pi) & \text{for all } t > 0\,, \end{cases} \tag{2}$$

(a) *(1 point).* Prove that the sequence $\{\varphi_N\}_{N\in\mathbb{N}}$ defined as $\varphi_N = \sum_{n=-N}^{N} a_n e^{inx}$ is a Cauchy sequence in $C^0[-\pi,\pi]$ and deduce that $\varphi \in C^0[-\pi,\pi]$.

(b) *(3 points).* Write $u(t,x) = \sum_{n=-\infty}^{\infty} b_n(t) e^{inx}$ and find from (2) an ODE satisfied formally for $b_n(t)$ in terms of a_n. Finally, compute explicitly $b_n(t)$.

(c) *(4 points).* Prove that the solution u found in the previous point is such that $u, \partial_t u, \partial_{xx} u \in C^0((0,\infty) \times [-\pi,\pi))$ and u solves (2) and that $u(t,x) \to 0$ as $t \to 0^+$ uniformly in x.

Hint: Recall that the solution of $\dot{y} = cy + d$ with $c, d \in \mathbb{R}$ and $y(0) = 0$ is $y(t) = \frac{d}{c}[e^{ct} - 1]$.

Question 15: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function such that

$$f(x) = \max\{0, \cos(x)\}$$

Denote with $\{a_n\}$ and $\{b_n\}$ the real Fourier coefficients of f, i.e.

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(a) (*5 points*). Compute the Fourier series of f

(b) (*1 point*). Compute the value of the series

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2 - 1},$$

Hint: $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$.

Question 16: *This question is worth 6 points.*

$\square_0$ $\square_1$ $\square_2$ $\square_3$ $\square_4$ $\square_5$ $\square_6$

(a) (*2 points*). Let $g \in C_c^\infty(\mathbb{R})$. Give a formula of $\widehat{g^{(k)}}(\xi)$ in terms of $\hat{g}(\xi)$ and prove it for $k = 1$, where $g^{(k)}$ denote the k-th derivative.

(b) (*4 points*). Is there a non-zero function $f \in \mathcal{S}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \xi^k f(\xi)\, d\xi = 0 \qquad \text{for every } k \in \mathbb{N}\,?$$

Hint: for (b) consider $g \in \mathcal{S}(\mathbb{R})$ such that $f = \widehat{g}$.

3

Teacher : Maria Colombo
Analysis 4 (MATH-205)
21/06/2023
Duration : 180 minutes

Student 3

SCIPER : **999002**

Do not turn the page before the start of the exam. This document is double-sided, has 20 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- Documents, books, calculators and mobile phones are **not** allowed to be used during the exam.
- All personal belongings (including turned-off mobiles) must be stored next to the walls of the classroom.
- You are allowed to bring to the exam **a one sided, A4 paper with notes handwritten by you personally.**
- For the **multiple choice** questions, we give :
  +1 points if your answer is correct,
  0 points if you give no answer or more than one,
  0 points if your answer is incorrect.
- The answers to the open questions must be justified. The derivation of the results must be clear and complete.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 For every $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(x) = \frac{1}{x^2}\mathbb{1}_{[2^{-n},2^{-n+1}]}$. Which of the following statements is **true**?

- ☐ $\lim_n \int_{\mathbb{R}} f_n \, dx < \int_{\mathbb{R}} \liminf_n f_n \, dx$.
- ☐ $\lim_n \int_{\mathbb{R}} |f_n - \lim_n f_n| \, dx = 0$.
- ☐ $\lim_n \int_{\mathbb{R}} f_n \, dx = \int_{\mathbb{R}} \lim_n f_n \, dx$.
- ☐ $\limsup_n \int_{\mathbb{R}} f_n \, dx > \int_{\mathbb{R}} \limsup_n f_n \, dx$.

Question 2 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : (0,1) \to \mathbb{R}$. Which of the following is **true**?

- ☐ Let $M \in \mathbb{N}$, if $f_n(x) \geq 0$ for any $x \in (0,1)$ and for any $n \leq M$, then
$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$
- ☐ Let $c \in \mathbb{R}$, if $f_n(x) \leq c$ for any $x \in (0,1)$ and for any $n \in \mathbb{N}$ then
$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$
- ☐ If $f_n(x) \leq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then
$$\liminf_{n\to\infty} \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_{n\to\infty} f_n(x)dx$$
- ☐ If $f_n(x) \geq 0$ for any $x \in (0,1)$ and $n \in \mathbb{N}$, then
$$\limsup_{n\to\infty} \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_{n\to\infty} f_n(x)dx$$

Question 3 Consider $f_n(x) = x^{-1/3}\mathbb{1}_{[0,1/n]}$. Then $f_n \to 0$

- ☐ in L^2 but not in L^4.
- ☐ in L^4 but not in L^∞.
- ☐ in L^∞.
- ☐ in L^1 but not in L^2.

Question 4 Let us consider the following PDE

$$\partial_x u - u\partial_y u = 0\,, \tag{1}$$

for $u : \mathbb{R}^2 \to \mathbb{R}$. Assume that v, w solve (1). Which of the following statement is **true**?

- ☐ For any v solution of (1) and any $\lambda > 0$, $v_\lambda(x,y) = \lambda v(\lambda x, y)$ solves (1).
- ☐ For any v, w solutions of (1), $v + w$ solves (1).
- ☐ $v(x,y) = x + xy + y^3 + x^2y$ is a solution of (1).
- ☐ For any v, w solutions of (1), $v - w$ solves (1).

Question 5 Let $f(x) = x^2 e^{-x^2}$, and denote with $\hat{f}(\xi)$ its Fourier Transform. Which of the following statements is **true**?

- [ ] $\lim_{\xi\to\pm\infty} \hat{f}(\xi) = 1$
- [ ] $\hat{f}(\xi) \in L^1 \cap L^\infty$
- [ ] $\hat{f}(\xi) \in L^2(\mathbb{R})$ but $\hat{f}(\xi) \notin L^1(\mathbb{R})$
- [ ] $\hat{f}(\xi)$ has compact support in $\mathbb{R}$

Question 6 Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $\hat{f}(\xi) = |\xi|^{5/2} \mathrm{e}^{-|\xi|^2/4}, \hat{g}(\xi) = \mathrm{e}^{-|\xi|^2/4}$. Which of the following is **true**?

- [ ] $g \notin \mathcal{S}(\mathbb{R})$.
- [ ] $f * g \notin L^\infty(\mathbb{R})$
- [ ] f and g have compact support.
- [ ] $f \notin \mathcal{S}(\mathbb{R})$.

Question 7 Which of the following sets has infinite outer measure?

- [ ] $\bigcup_{n=1}^{\infty} \left(2^n, \frac{2^{2n}+1}{2^n}\right)$
- [ ] $\bigcup_{n=1}^{\infty} \left(n-3, \frac{(n-1)^3}{n^2}\right)$
- [ ] $\bigcup_{n=1}^{\infty} \left(-\cos^2(\frac{1}{n}), \sin^2(\frac{1}{n})\right)$
- [ ] $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$

Question 8 Let $f \in C(\mathbb{R}/\mathbb{Z})$ be a 1-periodic function and let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$ be its real Fourier coefficents, namely $a_n = 2\int_0^1 f(x)\cos(2\pi nx)dx$, $b_n = 2\int_0^1 f(x)\sin(2\pi nx)dx$.
Assume that $f(x+1/2) = -f(x)$ for all x. Which of the following is **true**?

- [ ] $a_0 = 0$ and $a_{2m} = b_{2m} = 0$ for all $m = 1, 2, \ldots$.
- [ ] None of the others is true.
- [ ] $a_{2m+1} = b_{2m+1} = 0$ for all $m = 1, 2, \ldots$.
- [ ] $a_n = b_n = 0$ for all $n = 1, 2, \ldots$.

Question 9 Let $f, g \in \mathcal{S}(\mathbb{R})$ and $\hat{f}, \hat{g}$ be their Fourier transform. Which of the following inequality is **true**?

- [ ] $\|f * g\|_{L^2} \le \|\hat{f}\|_{L^2} \|\hat{g}\|_{L^2}$
- [ ] $\|f * g\|_{L^2} \le \|\hat{f}\|_{L^\infty} \|\hat{g}\|_{L^\infty}$
- [ ] $\|f * g\|_{L^2} \le \|\hat{f}\|_{L^\infty} \|\hat{g}\|_{L^2}$
- [ ] $\|f * g\|_{L^2} \le \|\hat{f}\|_{L^\infty} \|\hat{g}\|_{L^1}$

Question 10 Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \ge 0$. Which of the following statements is **true**?

- [ ] If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$.
- [ ] If $f, g \in L^1(\mathbb{R})$ then $fg \in L^1(\mathbb{R})$.
- [ ] If $f \in L^3(\mathbb{R})$ then $f^{3/2} \in L^2(\mathbb{R})$.
- [ ] If $f \in L^2(\mathbb{R})$ then $f \in L^1(\mathbb{R})$.

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 11: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Let $A = (1, \infty) \times [0, \pi/2]$, $n \in \mathbb{N}$ and $f_n : A \to \mathbb{R}$ defined as

$$f_n(x, y) = \frac{x + \cos(2y)}{(x^2 + 1 + \frac{y}{n})^2}.$$

Compute $\lim_{n\to\infty} f_n(x, y)$ and show that

$$\lim_{n\to\infty} \int_A f_n(x, y) dx\, dy = \frac{\pi}{8}.$$

Question 12: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Let $f \in L^1(\mathbb{R})$. Then, prove the following:

(a) *(2 points).*

$$\lim_{n\to\infty} \int_n^\infty |f(x)|dx = 0\,.$$

(b) *(1 point).*

$$\liminf_{x\to\infty} |f(x)| = 0\,.$$

(c) *(2 points).* There exists a function $f \in L^1(\mathbb{R})$ such that

$$\limsup_{x\to\infty} |f(x)| = \infty\,.$$

Question 13: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5

Consider two non-negative simple functions $f, g : [0,1] \to \mathbb{R}$, namely $f(x) = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i}(x)$, $g(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{C_i}(x)$ where $N \in \mathbb{N}$ and $a_i \geq 0\,, c_i \geq 0$, $C_i, A_i \subset [0,1]$ are measurable sets for any $i = 1, \ldots, N$. We associate to f, g the two functions

$$F(x) = \sum_{i=1}^{N} a_i \mathbb{1}_{[0,m(A_i)]}(x), \qquad G(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{[0,m(C_i)]}(x),$$

Prove the following:

(a) *(3 points).*

$$\int_0^1 |f(x)|dx = \int_0^1 |F(x)|dx$$
$$\int_0^1 f(x)g(x)dx \leq \int_0^1 F(x)G(x)dx$$

(b) *(2 points).* Finally, for $N = M = 1$ and $A_1 \cap C_1 = \emptyset$ prove that

$$\int_0^1 |F(x) - G(x)|dx \leq \int_0^1 |f(x) - g(x)|dx.$$

Question 14: *This question is worth 8 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $\sum_{n=-\infty}^{\infty}|a_n|<\infty$ and define $\varphi(x)=\sum_{n=-\infty}^{\infty}a_n e^{inx}$. Consider the PDE

$$\begin{cases} \partial_t u = 2\partial_{xx}u + u - \varphi \qquad (t,x)\in(0,\infty)\times(-\pi,\pi)\,, \\ u(0,x)\equiv 0 \qquad \text{for all } x\in[-\pi,\pi) \\ u(t,-\pi)=u(t,\pi) \qquad \text{for all } t>0\,, \end{cases} \tag{2}$$

(a) *(1 point).* Prove that the sequence $\{\varphi_N\}_{N\in\mathbb{N}}$ defined as $\varphi_N=\sum_{n=-N}^{N}a_n e^{inx}$ is a Cauchy sequence in $C^0[-\pi,\pi]$ and deduce that $\varphi\in C^0[-\pi,\pi]$.

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☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6

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☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

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Hint: for (b) consider $g \in \mathcal{S}(\mathbb{R})$ such that $f = \widehat{g}$.