Analysis III - 203(d)

Winter Semester 2024

Session 10: November 21, 2024

Exercise 1

$$ReLU(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \end{cases}$$

$$ReLU^k(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^k & \text{if } 0 \leq x \end{cases} \quad \text{where } k \in \mathbb{N}_0$$

$$f(x) = \begin{cases} -x-2 & \text{if } -2 < x < -1 \\ -x^2 - 2x - 1 & \text{if } -1 < x < 0 \\ x^2 - 2x + 1 & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$j(x) = \begin{cases} x & \text{if } x < 0 \\ 1 + x & \text{if } 0 \leq x \end{cases}.$$

Here, employ the definition of distributional derivatives and check against the formula for piecewise differentiable functions as obtained during lecture.

Solution 1

- *For the ReLU function, we have for any* $\phi \in \mathcal{D}(\mathbb{R})$,

$$\langle ReLU', \phi \rangle = -\langle ReLU, \phi' \rangle = -\int_{-\infty}^{\infty} ReLU(x)\phi'(x)dx = -\int_0^{\infty} x\phi'(x)dx = \int_0^{\infty} \phi(x)dx = \langle H, \phi \rangle,$$

  where H is the Heaviside step function. Thus, we conclude that $ReLU' = H$ in the sense of distributions.

- *For $k > 1$, we find for any $\phi \in \mathcal{D}(\mathbb{R})$,*

$$\begin{aligned} \langle (ReLU^k)', \phi \rangle &= -\langle ReLU^k, \phi' \rangle = -\int_{-\infty}^{\infty} ReLU^k(x)\phi'(x)dx \\ &= -\int_0^{\infty} x^k \phi'(x)dx = \int_0^{\infty} kx^{k-1}\phi(x)dx = \langle kReLU^{k-1}, \phi \rangle. \end{aligned}$$

  Therefore, we conclude that $(ReLU^k)' = kReLU^{k-1}$ in the sense of distributions.

- *For the function f, we have for any $\phi \in \mathcal{D}(\mathbb{R})$,*

$$
\begin{aligned}
\langle f', \phi \rangle = -\langle f, \phi' \rangle &= \int_{-2}^{-1} (x+2)\phi'(x)dx \\
&\quad + \int_{-1}^{0} (x^2+2x+1)\phi'(x)dx \\
&\quad - \int_{0}^{1} (x^2-2x+1)\phi'(x)dx \\
&\quad + \int_{1}^{2} (x-2)\phi'(x)dx \\
&= \phi(-1) - \int_{-2}^{-1} \phi(x)dx \\
&\quad + \phi(0) - \int_{-1}^{0} 2(x+1)\phi(x)dx \\
&\quad + \phi(0) + \int_{0}^{1} 2(x-1)\phi(x)dx \\
&\quad + \phi(1) - \int_{1}^{2} \phi(x)dx \\
&= \int_{-2}^{-1} (-1)\phi(x)dx + \int_{-1}^{0} (-2(x+1))\phi(x)dx \\
&\quad + \int_{0}^{1} 2(x-1)\phi(x)dx + \int_{1}^{2} (-1)\phi(x)dx + \phi(-1) + 2\phi(0) + \phi(1) \\
&= \langle g + \delta_{-1} + 2\delta_0 + \delta_1, \phi \rangle,
\end{aligned}
$$

where the function $g : \mathbb{R} \to \mathbb{R}$ is defined as

$$
g(x) = \begin{cases} -1 & \text{if } -2 < x \leq -1 \\ -2(x+1) & \text{if } -1 < x \leq 0 \\ 2(x-1) & \text{if } 0 < x \leq 1 \\ -1 & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases} .
$$

Hence, we conclude that $f' = g + \delta_{-1} + 2\delta_0 + \delta_1$. Note that this corresponds exactly to the formula found during the lecture.

- *For the function j, we have for any $\phi \in \mathcal{D}(\mathbb{R})$,*

$$
\langle j', \phi \rangle = -\langle j, \phi' \rangle = -\int_{-\infty}^{0} x\phi'(x)dx - \int_{0}^{\infty} (1+x)\phi'(x)dx
$$

$$= \int_{-\infty}^{0} \phi(x)dx + \int_{0}^{\infty} \phi(x)dx + \phi(0)$$
$$= \int_{-\infty}^{\infty} \phi(x)dx + \phi(0).$$

We conclude that $j' = 1 + \delta_0$. Note that we can also write $j(x) = x + H(x)$, where H is the heaviside function.

Exercise 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with period 1 and*

$$f(x) = x, \qquad 0 < x < 1.$$

Find the distributional derivative.

Solution 2 *For a test function $\phi \in \mathcal{D}(\mathbb{R})$, we have*

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle = -\sum_{n\in\mathbb{Z}} \int_{n}^{n+1} (x-n)\phi'(x)dx,$$

Here, we have used that the function looks like $x - n$ over each interval $(n, n-1)$: over the interval from n to $n+1$, it looks like a linear function with slope 1 that equals 0 at $x = n$. Now, use integration by parts:

$$-\sum_{n\in\mathbb{Z}} \int_{n}^{n+1} (x-n)\phi'(x)dx = -\sum_{n\in\mathbb{Z}} \left[\phi(n+1) - \int_{n}^{n+1} \phi(x)dx\right]$$
$$= -\sum_{n\in\mathbb{Z}} \phi(n+1) + \sum_{n\in\mathbb{Z}} \int_{n}^{n+1} \phi(x)dx$$
$$= -\sum_{n\in\mathbb{Z}} \phi(n) + \sum_{n\in\mathbb{Z}} \int_{n}^{n+1} \phi(x)dx$$
$$= -\sum_{n\in\mathbb{Z}} \phi(n) + \int_{-\infty}^{\infty} \phi(x)dx.$$

On a technical note, we notice that the sum only involves a finite number of terms and that the integral is finite, too, because ϕ is smooth with compact support. Hence, we have the distributional derivative equals the constant function 1 minus the Dirac comb.

$$f' = 1 - \Delta_1.$$

We observe that this accounts for the fact that f is a piecewise linear function with slope equal 1 on the intervals and with jumps -1 at each of the interval boundaries.

Exercise 3 *Compute the Fourier coefficients of the function f that has period $T = 2\pi$ and satisfies*

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & \pi < x \le 2\pi \end{cases}$$

Solution 3 *We notice that the period T equals $T = 2\pi$. We have defined the first coefficient to be just the average:*

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}.$$

For the other coefficients, we use the formulas from the lecture:

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{2\pi n x}{T}\right)\,dx \\
&= \frac{2}{2\pi}\int_0^{\pi} \cos\left(\frac{2\pi n x}{T}\right)\,dx \\
&= \frac{2}{2\pi}\left[\frac{T}{2\pi n}\sin\left(\frac{2\pi n x}{T}\right)\right]_{x=0}^{x=\pi} \\
&= \frac{2}{2\pi}\frac{T}{2\pi n}\left(\sin\left(\frac{2\pi n \pi}{T}\right) - \sin(0)\right) \\
&= \frac{2}{2\pi}\frac{2\pi}{2\pi n}\left(\sin\left(n\pi\right) - \sin(0)\right) \\
&= 0
\end{aligned}$$

because $\sin$ *equals zero at integer multiples of π. For the sine mode coefficients, we calculate similarly:*

$$\begin{aligned}
b_n &= \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx \\
&= \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{x=0}^{x=\pi} \\
&= -\frac{1}{\pi}\frac{1}{n}\left[\cos(n\pi) - \cos(0)\right] \\
&= -\frac{1}{n\pi}\left[(-1)^n - 1\right] \\
&= \frac{1}{n\pi}\begin{cases} 0 & \textit{if } n \textit{ is even} \\ 2 & \textit{if } n \textit{ is odd} \end{cases}
\end{aligned}$$

Thus we find

$$\frac{a_0}{2} = 0.5, \qquad a_1 = a_2 = \cdots = 0, \tag{1}$$

and either $b_n = 0$ if n is even and $b_n = \frac{2}{n\pi}$ if n is odd.

Exercise 4 *Compute the Fourier coefficients of the following functions, which have period $T = 1$ and have the given values over the interval $[0, 1)$:*

- $$f(x) = x^2$$
- $$g(x) = (1 - x)x$$
- $$h(x) = |\sin(2\pi x)|$$

Solution 4

- *To compute the Fourier coefficients of $f(x) = x^2$ over the interval $[0, 1)$ with a period of 1, use the formulas from the lecture:*

$$\frac{a_0}{2} = \int_0^1 f(x)dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

For other coefficients, we apply integration by parts twice, and we use that $\sin$ *vanishes at multiples of π:*

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^1 x^2 \cos(2\pi nx)dx \\
&= \frac{2}{T}\left[x^2 \frac{\sin(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} - \frac{2}{T}\int_0^1 \frac{2x\sin(2\pi nx)}{2\pi n}dx \\
&= \frac{2}{T}\left[x^2 \frac{\sin(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} - \frac{4}{T}\int_0^1 x\frac{\sin(2\pi nx)}{2\pi n}dx \\
&= \frac{2}{T}\left[x^2 \frac{\sin(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} - \frac{4}{T}\left[x\frac{-\cos(2\pi nx)}{4\pi^2 n^2}\right]_{x=0}^{x=1} + \frac{4}{T}\int_0^1 \frac{-\cos(2\pi nx)}{4\pi^2 n^2}dx \\
&= \frac{2}{T}\left[x^2 \frac{\sin(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} - \frac{4}{T}\left[x\frac{-\cos(2\pi nx)}{4\pi^2 n^2}\right]_{x=0}^{x=1} + \frac{4}{T}\left[\frac{-\sin(2\pi nx)}{8\pi^3 n^3}\right]_{x=0}^{x=1} \\
&= \frac{1}{\pi^2 n^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^1 x^2 \sin(2\pi nx)dx \\
&= \frac{2}{T}\left[-x^2 \frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{2}{T}\int_0^1 \frac{2x\cos(2\pi nx)}{2\pi n}dx \\
&= \frac{2}{T}\left[-x^2 \frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{4}{T}\int_0^1 \frac{x\cos(2\pi nx)}{2\pi n}dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{T}\left[-x^2\frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{4}{T}\left[x\frac{\sin(2\pi nx)}{4\pi^2n^2}\right]_{x=0}^{x=1} - \frac{4}{T}\int_0^1 \frac{\sin(2\pi nx)}{4\pi^2n^2}dx \\
&= \frac{2}{T}\left[-x^2\frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{4}{T}\left[x\frac{\sin(2\pi nx)}{4\pi^2n^2}\right]_{x=0}^{x=1} + \frac{4}{T}\left[\frac{\cos(2\pi nx)}{8\pi^3n^3}\right]_{x=0}^{x=1} \\
&= -\frac{2}{T}\frac{1}{2\pi n} + \frac{4}{T}\frac{1}{8\pi^3n^3} - \frac{4}{T}\frac{1}{8\pi^3n^3} = -\frac{1}{\pi n}
\end{aligned}
$$

- *We notice that the function $g(x) = (1-x)x$ can be written $g(x) = x - f(x)$ over the interval $[0,1)$. In other words, g is the difference of the sawtooth function (which equals x over $[0,1)$ and repeats with period $T = 1$), and the periodic function $f(x)$ seen earlier.*

  We will only need to determine the Fourier coefficients of $\tilde{g}(x) = x$ since we already computed the Fourier coefficients of $f(x)$. We calculate the average

$$
\frac{a_0^{\tilde{g}}}{2} = \int_0^1 \tilde{g}(x)dx = \int_0^1 xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.
$$

  We calculate

$$
\begin{aligned}
a_n^{\tilde{g}} &= \frac{2}{T}\int_0^1 x\cos(2\pi nx)dx \\
&= \frac{2}{T}\left[x\frac{\sin(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} - \frac{2}{T}\int_0^1 \frac{\sin(2\pi nx)}{2\pi n}dx \\
&= \frac{2}{T}\left[x\frac{\sin(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{2}{T}\left[\frac{\cos(2\pi nx)}{4\pi^2n^2}\right]_{x=0}^{x=1} \\
&= \frac{2}{T}\frac{1}{2\pi n}\left(\sin(2\pi n) - \sin(0)\right) + \frac{2}{T}\frac{1}{4\pi^2n^2}\left(\cos(2\pi n) - \cos(0)\right) \\
&= 0
\end{aligned}
$$

  Lastly,

$$
\begin{aligned}
b_n^{\tilde{g}} &= \frac{2}{T}\int_0^1 x\sin(2\pi nx)dx \\
&= \frac{2}{T}\left[-x\frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{2}{T}\int_0^1 \frac{\cos(2\pi nx)}{2\pi n}dx \\
&= \frac{2}{T}\left[-x\frac{\cos(2\pi nx)}{2\pi n}\right]_{x=0}^{x=1} + \frac{2}{T}\left[\frac{\sin(2\pi nx)}{4\pi^2n^2}\right]_{x=0}^{x=1} \\
&= -\frac{2}{T}\frac{1}{2\pi n} = -\frac{1}{\pi n}
\end{aligned}
$$

  Now the Fourier coefficients of $g(x)$ are given by

$$
b_n = -\frac{1}{\pi n} - -\frac{1}{\pi n} = 0
$$

$$a_n = 0 - \frac{1}{\pi^2 n^2} = -\frac{1}{\pi^2 n^2}$$
$$\frac{a_0}{2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

- *Note that $h(x)$ is an even function. We first check that the sine modes are have coefficients equal zero.*

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T |\sin(2\pi x)|\sin(2\pi n x)dx \\ &= 2\int_0^1 |\sin(2\pi x)|\sin(2\pi n x)dx \\ &= 2\int_0^{\frac{1}{2}} \sin(2\pi x)\sin(2\pi n x)dx - 2\int_{\frac{1}{2}}^1 \sin(2\pi x)\sin(2\pi n x)dx. \end{aligned}$$

One possibility to proceed is via substitution in the second integral, setting $x = y + \frac{1}{2}$. We then find

$$\begin{aligned} &\int_{\frac{1}{2}}^1 \sin(2\pi x)\sin(2\pi n x)dx \\ &\quad = \int_0^{\frac{1}{2}} \sin(2\pi(x+0.5))\sin(2\pi n(x+0.5))dx \\ &\quad = \int_0^{\frac{1}{2}} \sin(2\pi x+\pi)\sin(2\pi n x+\pi n)dx = (-1)^{n+1}\int_0^{\frac{1}{2}} \sin(2\pi x)\sin(2\pi n x+\pi n)dx. \end{aligned}$$

In the last step, we have used the symmetry properties of the sine function. Hence

$$b_n = 2\left(1-(-1)^{n+1}\right)\int_0^{\frac{1}{2}} \sin(2\pi x)\sin(2\pi n x)dx.$$

If n is odd, then the last expression is already zero. Otherwise, if n is even, we want to show that last integral is zero. We use the sum formula:

$$\sin(2\pi x)\sin(2\pi n x) = \frac{1}{2}\cos(2\pi(n-1)x) - \frac{1}{2}\cos(2\pi(n+1)x).$$

With that observation, we find

$$\begin{aligned} b_n &= \frac{1}{2}\int_0^{\frac{1}{2}} \cos(2\pi(n-1)x) - \cos(2\pi(n+1)x)\ dx \\ &= \frac{1}{2}\left[\frac{\sin(2\pi(n-1)x)}{2\pi(n-1)} - \frac{\sin(2\pi(n+1)x)}{2\pi(n+1)}\right]_{x=0}^{x=\frac{1}{2}}. \end{aligned}$$

For $x = 0$, the two sine terms are already zero, and for $x = \frac{1}{2}$, the two sine terms are evaluated at integer multiples of π and hence equal zero, too. We conclude that $b_n = 0$.

Next, we calculate the average

$$\frac{a_0}{2} = \frac{1}{T}\int_0^1 |\sin 2\pi x| dx = \frac{2}{T}\int_0^{\frac{1}{2}} \sin 2\pi x dx = \frac{2}{T}\left[\frac{-1}{2\pi}\cos 2\pi x\right]_0^{\frac{1}{2}} = \frac{2}{\pi}.$$

The coefficients a_n for the cosine modes with $n \geq 1$ are:

$$\begin{aligned}
a_n &= \frac{4}{T}\int_0^{\frac{1}{2}} \sin 2\pi x \cos(2\pi n x) dx \\
&= \frac{4}{T}\int_0^{\frac{1}{2}} \frac{1}{2}\sin 2\pi(1-n)x + \frac{1}{2}\sin 2\pi(n+1)x dx \\
&= \frac{4}{T}\int_0^{\frac{1}{2}} -\frac{1}{2}\sin 2\pi(n-1)x + \frac{1}{2}\sin 2\pi(n+1)x dx \\
&= \left[\frac{\cos 2\pi(n-1)x}{\pi(n-1)}\right]_{x=0}^{x=\frac{1}{2}} + \left[-\frac{\cos 2\pi(n+1)x}{\pi(n+1)}\right]_{x=0}^{x=\frac{1}{2}} \\
&= \frac{(-1)^{n-1}}{\pi(n-1)} - \frac{1}{\pi(n-1)} - \frac{(-1)^{n+1}}{\pi(n+1)} + \frac{1}{\pi(n+1)} \\
&= -\frac{1+(-1)^n}{\pi(n-1)} + \frac{1-(-1)^{n+1}}{\pi(n+1)} \\
&= -\frac{1+(-1)^n}{\pi(n-1)} + \frac{1+(-1)^n}{\pi(n+1)}
\end{aligned}$$

Therefore, when $n \geq 1$:

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ -\frac{2}{\pi(n-1)} + \frac{2}{\pi(n+1)}, & \text{if } n \text{ is even,} \end{cases}$$