

Analysis III - 203(d)

Winter Semester 2024

Session 9: November 14, 2024

Exercise 1 *Show that*

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (1)$$

Hint: square the integral and convert it to polar/radial coordinates.

Solution 1 *Following the hint, we find*

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \quad (2)$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = 2\pi \int_0^{\infty} e^{-r^2} r dr \quad (3)$$

$$= \pi \int_0^{\infty} e^{-u} du = \pi [-e^u]_{u=0}^{u=\infty} = \pi, \quad (4)$$

where we have used Fubini's Theorem and the substitution $u = r^2$. Taking the square root of both sides gives the desired result.

Exercise 2 *We check whether the following are distributions or not.*

- *Show that T is a distribution, where*

$$T(\phi) = \int_{-1}^1 \phi(x) dx. \quad (5)$$

- *Show that T is a distribution, where*

$$T(\phi) = \int_{-\infty}^{\infty} \phi(x) dx. \quad (6)$$

- *Show that S is not a distribution, where*

$$S(\phi) = \int_0^1 |\phi(x)| dx. \quad (7)$$

Solution 2 The definition in (5) is clearly linear in ϕ as the integral is linear and finite for any $\phi \in \mathcal{D}$ since continuous functions on compact sets are bounded. In particular, we have

$$|T(\phi)| = \left| \int_{-1}^1 \phi(x) dx \right| \leq \int_{-1}^1 |\phi(x)| dx \leq \sup_{x \in [-1,1]} |\phi(x)| \int_{-1}^1 dx = 2 \sup_{x \in [-1,1]} |\phi(x)|, \quad (8)$$

which proves the continuity condition.

The definition in (6) is also linear in ϕ for the same reason as before. To show that the integral is finite for any $\phi \in \mathcal{D}$, we use the fact that ϕ is compactly supported and hence bounded. In particular, there exists for any $\phi \in \mathcal{D}$ a compact interval $[a, b] \subset \mathbb{R}$ such that $\text{supp}(\phi) \subset [a, b]$. Then we have

$$|T(\phi)| = \left| \int_{-\infty}^{\infty} \phi(x) dx \right| \leq \int_a^b |\phi(x)| dx \leq \sup_{x \in [a,b]} |\phi(x)| \int_a^b dx = (b-a) \sup_{x \in [a,b]} |\phi(x)| < \infty, \quad (9)$$

which shows that the integral is finite for any $\phi \in \mathcal{D}$. Conversely, for any compact interval $[a, b] \in \mathbb{R}$ and $\phi \in \mathcal{D}$, such that $\text{supp}(\phi) \subset [a, b]$, we have

$$|T(\phi)| = \left| \int_{-\infty}^{\infty} \phi(x) dx \right| \leq \int_a^b |\phi(x)| dx \leq \sup_{x \in [a,b]} |\phi(x)| \int_a^b dx = (b-a) \sup_{x \in [a,b]} |\phi(x)|, \quad (10)$$

which shows the continuity condition.

The definition in (7) is not a distribution since it is not linear in ϕ . There are many different ways to see this. For example, consider $\phi \in \mathcal{D}$ such that $\phi(x) \geq 0$ for all $x \in \mathbb{R}$. We have seen such a function in the lecture. Set $\psi = -\phi \in \mathcal{D}$. Then we have

$$S(\phi + \psi) = \int_0^1 |\phi(x) + \psi(x)| dx = \int_0^1 0 dx = 0, \quad (11)$$

but

$$S(\phi) + S(\psi) = \int_0^1 |\phi(x)| dx + \int_0^1 |\psi(x)| dx = 2 \int_0^1 |\phi(x)| dx > 0. \quad (12)$$

Exercise 3 Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an integrable function. Whenever $a > 0$, show that

$$T(\phi) = a \langle f, \phi(a \cdot) \rangle \quad (13)$$

is a distribution.

Solution 3 With the substitution $u = ax$, hence $du = a dx$, we find

$$\int f(u) \phi(u/a) \frac{1}{a} du = \int f(ax) \phi(x) dx. \quad (14)$$

Clearly, the function $g(x) := f(ax)$ is integrable. That means

$$T(\phi) = \int g(x)\phi(x) \, dx, \quad (15)$$

which is a distribution.

Exercise 4 Find the distributional derivative of the function

$$f(x) = |x|. \quad (16)$$

Solution 4 For any test function $\phi \in \mathcal{D}$, we have

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} |x|\phi'(x) \, dx \quad (17)$$

$$= -\int_0^{\infty} x\phi'(x) \, dx + \int_{-\infty}^0 x\phi'(x) \, dx \quad (18)$$

$$= -[x\phi(x)]_0^{\infty} + \int_0^{\infty} \phi(x) \, dx + [x\phi(x)]_{-\infty}^0 - \int_{-\infty}^0 \phi(x) \, dx \quad (19)$$

$$= \int_0^{\infty} \phi(x) \, dx - \int_{-\infty}^0 \phi(x) \, dx, \quad (20)$$

where we have used that ϕ is compactly supported and hence the boundary terms vanish. We define the function g by

$$g(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \quad (21)$$

Apparently,

$$\langle f', \phi \rangle = \langle g, \phi \rangle. \quad (22)$$

Note that we can also write this in terms of the Heaviside step function,

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \quad (23)$$

One easily sees that $g(x) = 2H(x) - 1$ is an alternative way of writing the distributional derivative.

Exercise 5 Consider the function with period 2 that satisfies

$$f(x) = \begin{cases} -x & \text{if } -1 < x \leq 0 \\ x & \text{if } 0 < x \leq 1 \end{cases} \quad (24)$$

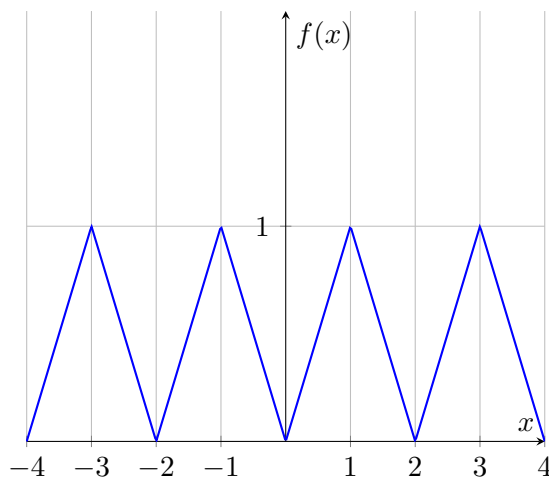


Figure 1: Plot of the periodic function f .

- Draw a plot of this function from -4 to 4 .
- Is this function differentiable? What is the first distributional derivative of this function?
- What is the second distributional derivative of this function?

Solution 5 The solution is plotted in Figure 1. The function is not differentiable at $x \in \mathbb{Z}$ since the left and right derivatives do not agree: for example, at $x = 0$ the derivative from the left equals -1 but the derivative from the right equals 1 .

It remains to compute the distributional derivative. If we look at the drawing we see that the function is continuous and differentiable over the intervals between integers. Over the interval $(n, n+1)$, the derivative is either 1 if n is even or it is -1 if n is odd. Moreover, the function does not jump. This gives us an idea what the distributional derivative should look like:

$$f'(x) = \begin{cases} 1 & \text{if } x \in (n, n+1), n \text{ even,} \\ -1 & \text{if } x \in (n, n+1), n \text{ odd.} \end{cases} \quad (25)$$

Let us formalize this intuition. We use the definition of distributional derivative. For any test function $\phi \in \mathcal{D}$, we have by definition of the distributional derivative that

$$\begin{aligned} \langle f', \phi \rangle &= -\langle f, \phi' \rangle \\ &= -\int_{-\infty}^{\infty} f(x) \phi'(x) \, dx \\ &= -\sum_{\substack{n \in \mathbb{Z} \\ n \text{ even}}} \int_n^{n+1} f(x) \phi'(x) \, dx - \sum_{\substack{n \in \mathbb{Z} \\ n \text{ odd}}} \int_n^{n+1} f(x) \phi'(x) \, dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{n \in \mathbb{Z} \\ n \text{ even}}} \int_n^{n+1} \phi(x) \, dx - f(n+1)\phi(n+1) - f(n)\phi(n) \\
&\quad - \sum_{\substack{n \in \mathbb{Z} \\ n \text{ odd}}} \int_n^{n+1} \phi(x) \, dx - f(n+1)\phi(n) - f(n)\phi(n) \\
&= \sum_{\substack{n \in \mathbb{Z} \\ n \text{ even}}} \int_n^{n+1} \phi(x) \, dx - \sum_{\substack{n \in \mathbb{Z} \\ n \text{ odd}}} \int_n^{n+1} \phi(x) \, dx \\
&\quad + \sum_{\substack{n \in \mathbb{Z} \\ n \text{ even}}} -f(n+1)\phi(n+1) - f(n)\phi(n) - \sum_{\substack{n \in \mathbb{Z} \\ n \text{ odd}}} -f(n+1)\phi(n) - f(n)\phi(n).
\end{aligned}$$

We simplify this. The point evaluations cancel out each other. Then

$$\langle f', \phi \rangle = \sum_{n \in \mathbb{Z}} \int_n^{n+1} (-1)^n \phi(x) \, dx. \quad (26)$$

This is exactly the form of f' as proposed above. It also corresponds to our observations in the lecture.

For the second distributional derivative, we again use the definition and simplify:

$$\langle f'', \phi \rangle = -\langle f', \phi' \rangle \quad (27)$$

$$= - \sum_{n \in \mathbb{Z}} (-1)^n \int_n^{n+1} \phi'(x) \, dx \quad (28)$$

$$= - \sum_{n \in \mathbb{Z}} (-1)^n (\phi(n+1) - \phi(n)) \quad (29)$$

$$= \sum_{n \in \mathbb{Z}} (-1)^{n+1} (\phi(n+1) - \phi(n)). \quad (30)$$

As we now carefully observe for term in the sum: when n is odd, the summand is $\phi(n+1) - \phi(n)$, and when n is even, then the summand is $-\phi(n+1) + \phi(n)$. In other words, for every even integer n we receive two positive Dirac deltas, and for every odd integer n we receive two negative Dirac deltas. Thus,

$$\langle f'', \phi \rangle = \sum_{n \in \mathbb{Z}} (-1)^n 2\delta_n. \quad (31)$$

This corresponds to the jumps of the distributional derivative f' . we jump (up) by 2 at every even integer and we jump (down) by -2 at every odd integer.