Analysis III - 203(d)

Winter Semester 2024

Session 14: December 19, 2025

Exercise 1 *Given the following functions over an interval* $[0, 1)$,

(a) $f(x) = x$

(b) $g(x) = x^2$

(c) $h(x) = e^x$

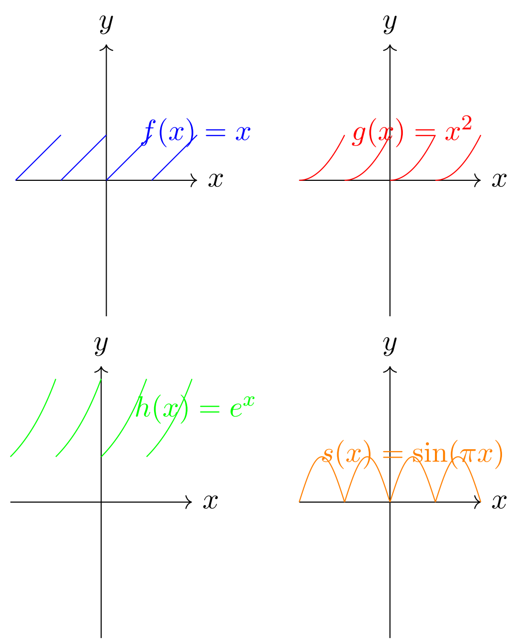
(d) $s(x) = sin(\pi x)$

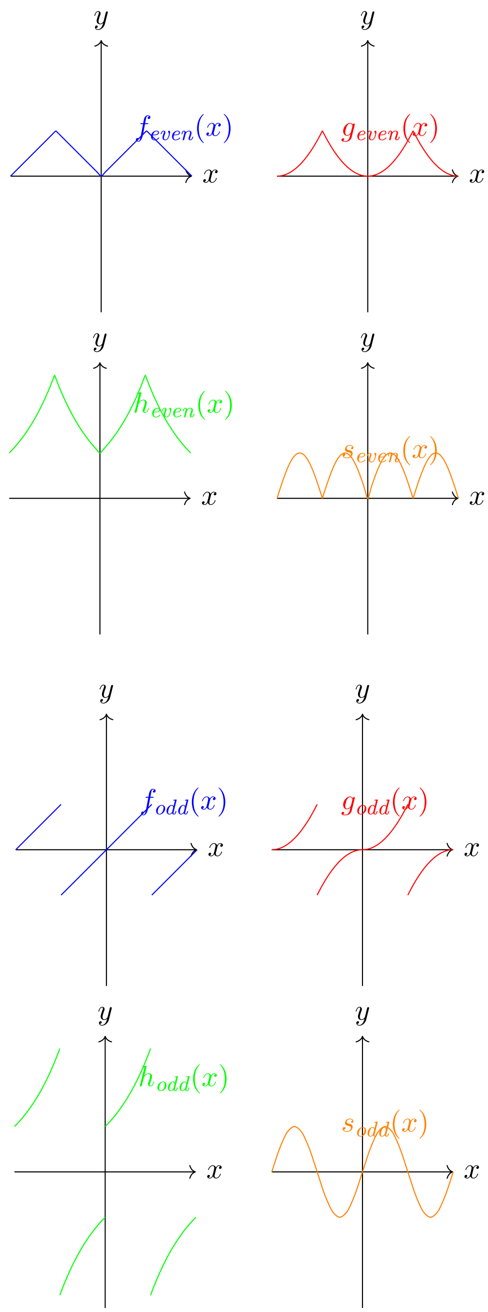
sketch their extension to

- *a periodic function with period* 1,
- *an even periodic function with period* 2,
- *an odd periodic function with period* 2.

State the formulas for the even and odd 2*-periodic extensions over the interval* $[-1, 1]$.

Solution 1 *We begin with these plots:*





We state the formulas for the even and odd extensions of period 2*. Over* $[-1,1]$*, define the even extensions:*

$$f_{even}(x) = |x|,$$
$$g_{even}(x) = x^2,$$

$$h_{even}(x) = e^{|x|},$$
$$s_{even}(x) = |\sin(\pi x)|,$$

and we define the odd extensions:

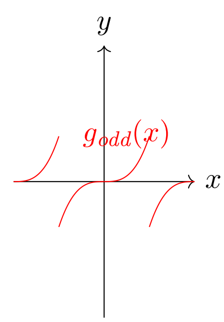
$$f_{odd}(x) = x,$$
$$g_{odd}(x) = \begin{cases} x^2 & x \in [0,1) \\ -(-x)^2 = -x^2 & x \in [-1,0) \end{cases}$$
$$h_{odd}(x) = \begin{cases} e^x & x \in [0,1) \\ -e^{-x} & x \in [-1,0) \end{cases},$$
$$s_{odd}(x) = \sin(\pi x).$$

Exercise 2 *Consider the function*

$$f : [0,1] \to \mathbb{R}, \quad x \mapsto x^3.$$

Extend this to an odd function with period $T = 2$. Sketch the graph of that function from -2 to 2. Compute its Fourier coefficients in standard form. Compute the complex Fourier coefficients.

Solution 2 *We first sketch the odd extensions of that function:*



The odd extension of period 2 is given by

$$f_{odd}(x) = x^3, \quad x \in [-1,1].$$

For the Fourier coefficients in standard form, we note that $a_n = 0, n \in \mathbb{N}$ since the function is odd by construction. For the sine-terms, we have

$$b_n = \int_{-1}^{1} f_{odd}(x) \sin(\pi n x)\ dx = 2\int_0^1 x^3 \sin(\pi n x)\ dx = \frac{2(-1)^n(6-\pi^2 n^2)}{\pi^3 n^3}.$$

We directly compute the integral via repeated integration by parts:

$$\int_0^1 x^3 \sin(\pi n x)\ dx = \left[x^3 \tfrac{(-1)}{\pi n} \cos(\pi n x)\right]_{x=0}^{x=1} + 3\tfrac{(-1)}{\pi n} \int_0^1 x^2 \cos(\pi n x)\ dx$$

$$
\begin{aligned}
&= \tfrac{(-1)}{\pi n}\cos(\pi n) + 3\tfrac{(-1)}{\pi n}\int_0^1 x^2\cos(\pi n x)\ dx \\
&= -\tfrac{(-1)^n}{\pi n} + 3\tfrac{(-1)}{\pi n}\int_0^1 x^2\cos(\pi n x)\ dx.
\end{aligned}
$$

$$
\int_0^1 x^2\cos(\pi n x)\ dx = \left[x^2\tfrac{1}{\pi n}\sin(\pi n x)\right]_{x=0}^{x=1} + 2\tfrac{1}{\pi n}\int_0^1 x\sin(\pi n x)\ dx = 2\tfrac{1}{\pi n}\int_0^1 x\sin(\pi n x)\ dx.
$$

$$
\begin{aligned}
\int_0^1 x\sin(\pi n x)\ dx &= \left[x\tfrac{(-1)}{\pi n}\cos(\pi n x)\right]_{x=0}^{x=1} + \tfrac{(-1)}{\pi n}\int_0^1 \cos(\pi n x)\ dx \\
&= \left[x\tfrac{(-1)}{\pi n}\cos(\pi n x)\right]_{x=0}^{x=1} + \tfrac{1}{\pi^2 n^2}\left[\sin(\pi n x)\right]_{x=0}^{x=1} \\
&= (-1)^n\tfrac{(-1)}{\pi n}.
\end{aligned}
$$

Putting all this together, we obtain

$$
\begin{aligned}
\int_0^1 x^3\sin(\pi n x)\ dx &= -\tfrac{(-1)^n}{\pi n} + 3\tfrac{(-1)}{\pi n}2\tfrac{1}{\pi n}(-1)^n\tfrac{(-1)}{\pi n} \\
&= -\frac{(-1)^n}{\pi n} + \frac{6}{\pi^3 n^3}(-1)^n \\
&= (-1)^n\frac{\pi^2 n^2 - 6}{\pi^3 n^3}.
\end{aligned}
$$

For the complex Fourier coefficients, we find for $n \in \mathbb{N}$*:*

$$
\begin{aligned}
c_0 &= a_0 = 0, \\
c_n &= \frac{1}{2}(a_n - ib_n) = \frac{i(-1)^{n+1}(\pi^2 n^2 - 6)}{\pi^3 n^3}, \\
c_{-n} &= \frac{1}{2}(a_n + ib_n) = \frac{i(-1)^n(\pi^2 n^2 - 6)}{\pi^3 n^3}.
\end{aligned}
$$

Exercise 3 *Suppose that*

$$
f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0\ , \\ 1-x & \text{if } 0 < x < 1\ , \\ 0 & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2} & \text{if } -1 < x < 1\ , \\ 0 & \text{otherwise.} \end{cases} \qquad h(x) = |x|.
$$

Compute the convolutions $u(x) = (f \star g)(x)$ *and* $v(x) = (g \star g)(x)$ *and* $w(x) = (g \star h)(x)$.

Solution 3 *We can compute these convolutions via direct computations or by results from Fourier analysis.*

- *This first one is the most difficult one. First, we note that*

$$g(x) = \frac{1}{2} 1_{[-1,1]}(x), \qquad 1_{[-1,1]}(x) = \begin{cases} 1 & x \in [-1,1], \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$f(x) = (x+1)1_{[-1,0]}(x) + (1-x)1_{(0,1]}(x).$$

Furthermore, we have for any $x, y \in \mathbb{R}$ that

$$1_{[-1,1]}(x-y) = 1_{[x-1,x+1]}(y).$$

Hence, we can write

$$(f \star g)(x) = \frac{1}{2} \int_{-\infty}^{\infty} f(y) 1_{[x-1,x+1]}(y) dy.$$

We think of this as a subinterval $[x-1, x+1]$ that moves over the real line, and we integrate f over this subinterval. Since f change its behavior three different times, the integral in the definition of $f \star g$ above will also change its behavior several times, depending on x.

We therefore use a case distinction.

1. *If $x < -2$, the integral is just over the region where f equals zero, and so $(f \star g)(x) = 0$.*
2. *If $-2 < x < -1$, then we only need to integrate f over $[-1, x+1] \subseteq [-1, 0]$. We find*

$$(f \star g)(x) = \frac{1}{2} \int_{-1}^{x+1} (y+1) dy = \frac{1}{4}(y+1)^2 \Big|_{y=-1}^{y=x+1} = \frac{1}{4}(x+2)^2.$$

3. *If $-1 < x < 0$, then we integrate f over $[-1, 0] \cup [0, x+1] \subseteq [-1, 1]$. We find*

$$\begin{aligned} (f \star g)(x) &= \frac{1}{2} \int_{-1}^{0} (y+1) dy + \frac{1}{2} \int_{0}^{x+1} (1-y) dy \\ &= \frac{1}{4}(y+1)^2 \Big|_{y=-1}^{y=0} - \frac{1}{4}(1-y)^2 \Big|_{y=0}^{y=x+1} = \frac{1}{4} - \left(\frac{x^2}{4} - \frac{1}{4} \right) = \frac{1}{2} - \frac{1}{4}x^2. \end{aligned}$$

4. *If $0 < x < 1$, then we integrate f over $[x-1, 0] \cup [0, 1] \subseteq [-1, 1]$. We find*

$$\begin{aligned} (f \star g)(x) &= \frac{1}{2} \int_{x-1}^{0} (y+1) dy + \frac{1}{2} \int_{0}^{1} (1-y) dy = \frac{1}{2} - \frac{1}{4}x^2 \\ &= \frac{1}{4}(y+1)^2 \Big|_{y=x-1}^{y=0} - \frac{1}{4}(1-y)^2 \Big|_{y=0}^{y=1} = \left(\frac{1}{4} - \frac{x^2}{4} \right) + \frac{1}{4} = \frac{1}{2} - \frac{1}{4}x^2. \end{aligned}$$

5. *If* $1 < x < 2$*, then we integrate* f *over* $[x-1, 1] \subseteq [0, 1]$*. We find*

$$(f \star g)(x) = \frac{1}{2}\int_{x-1}^{1} (1-y)dy = \frac{1}{4}(x-2)^2$$

6. *Lastly, if* $2 < x$*, the integral is just over the region where* f *equals zero, and so* $(f \star g)(x) = 0$*.*

- *Note that*

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(\omega/2)}{\omega/2}\right),$$
$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{\sin(\omega)}{\omega}.$$

Using the Convolution Theorem, we find

$$\hat{v}(\omega) = \sqrt{2\pi}(\hat{g}(\omega))^2 = \hat{f}(2\omega)$$

Using the Modulation Theorem, we find

$$v(x) = \frac{1}{2}\mathcal{F}^{-1}(2\hat{f}(2\omega)) = \frac{1}{2}f(\frac{x}{2})$$
$$= \frac{1}{4}\begin{cases} 2+x, & -2 \le x < 0, \\ 2-x, & 0 \le x < 2. \end{cases}$$

Alternatively, consider

$$(g \star g)(x) = \frac{1}{2}\int_{x-1}^{x+1} g(y)\ dy.$$

The g *in the integral has support* $[-1, 1]$*. Clearly, this integral equals* 0 *when* $x < -2$ *or when* $x > 2$ *because then* $(x-1, x+1)$ *and* $(-1, 1)$ *are disjoint. So it remains to consider the case* $-2 < x < 2$*. The integral equals (up to a factor of* $\frac{1}{2}$*, the length of the intersection of* $(x-1, x+1)$ *and* $(-1, 1)$*. To put this into a formula, it seems reasonable to distinguish whether* x *lies to the left or to the right of the origin. If* $-2 \le x \le 0$*, then*

$$\int_{x-1}^{x+1} g(y)\ dy = \int_{-1}^{x+1} g(y)\ dy = \frac{1}{2}((x+1) - (-1)) = \frac{1}{2}(x+2).$$

Hence

$$(g \star g)(x) = \frac{1}{4}(x+2).$$

If $0 \le x \le 2$*, then*

$$\int_{x-1}^{x+1} g(y)\ dy = \int_{x-1}^{1} g(y)\ dy = \frac{1}{2}((1) - (x-1)) = \frac{1}{2}(2-x).$$

Hence

$$(g \star g)(x) = \frac{1}{4}(2-x).$$

- *We first observe*

$$(g \star h)(x) = \frac{1}{2}\int_{x-1}^{x+1} |y| dy.$$

The absolute value function has two different regimes. From here, we make a case distinction, depending on whether $(x-1, x+1)$ *lies in one of the regimes or the other. If* $0 < x-1$*, which means* $1 < x$*, then* $(x-1, x+1)$ *lies within the positive real numbers and*

$$(g \star h)(x) = \frac{1}{2}\int_{x-1}^{x+1} y dy = x.$$

If $x+1 < 0$*, which means* $x < -1$*, then* $(x-1, x+1)$ *lies within the negative real numbers and*

$$(g \star h)(x) = \frac{1}{2}\int_{x-1}^{x+1} (-y) dy = -x.$$

The case $x-1 < 0 < x+1$*, which is* $-1 < x < 1$*, is more demanding. We split*

$$\begin{aligned}(g \star h)(x) &= \frac{1}{2}\int_{x-1}^{0} -y dy + \frac{1}{2}\int_{0}^{x+1} y dy \\ &= \frac{1}{2}\left[-\frac{1}{2}y^2\right]_{y=x-1}^{y=0} + \frac{1}{2}\left[\frac{1}{2}y^2\right]_{y=0}^{y=x+1} \\ &= \frac{1}{4}(x-1)^2 + \frac{1}{4}(x+1)^2 = \frac{1}{2}x^2 + \frac{1}{2}.\end{aligned}$$

Note: if you plot this function, it will look like a moving average of $|x|$ *that has been smoothed around* $x = 0$*.*

Exercise 4 *Suppose that* $f(x) = x^2$ *and that*

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } -1 < x < 1 \ , \\ 0 & \text{otherwise.} \end{cases} \qquad h(x) = \begin{cases} e^{-x} & \text{if } x > 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

Compute the convolutions $u(x) = (f \star g)(x)$ *and* $v(x) = (f \star h)(x)$ *.*

Solution 4 *We find by direct computation:*

$$\begin{aligned}(f \star g)(x) = (g \star f)(x) &= \frac{1}{2}\int_{-1}^{1}(x-y)^2 dy \\ &= \frac{1}{2}\left(\frac{1}{3}(x+1)^3 - \frac{1}{3}(x-1)^3\right) \\ &= \frac{1}{6}\left((x^3+3x^2+3x+1) - (x^3-3x^2+3x-1)\right) \\ &= \frac{1}{6}\left(6x^2+2\right) \\ &= x^2 + \frac{1}{3}.\end{aligned}$$

Similarly,

$$(f \star h)(x) = \int_0^\infty (x-y)^2 e^{-y} dy = \int_0^\infty (y-x)^2 e^{-y} dy.$$

We proceed here with integration by parts: first,

$$\begin{aligned}\int_0^\infty (y-x)^2 e^{-y} dy &= \left[-(y-x)^2 e^{-y}\right]_{y=0}^{y=\infty} - \int_0^\infty (-1)2(y-x)e^{-y} dy \\ &= x^2 + 2\int_0^\infty (y-x)e^{-y} dy.\end{aligned}$$

Then,

$$\begin{aligned}\int_0^\infty (y-x)e^{-y} dy &= \left[-(y-x)e^{-y}\right]_{y=0}^{y=\infty} - \int_0^\infty (-1)e^{-y} dy \\ &= -x + \int_0^\infty e^{-y} dy.\end{aligned}$$

Lastly,

$$\int_0^\infty e^{-y} dy = \left[-e^{-y}\right]_{y=0}^{y=\infty} = 1.$$

Thus, in total, we obtain

$$(f \star h)(x) = x^2 + 2\left(-x+1\right) = x^2 - 2x + 2.$$

Exercise 5 *We have discussed solutions to the differential equation*

$$-\Delta u(x) + k^2 u(x) = e^{-|x|}, \quad x \in \mathbb{R}.$$

- *Verify that, in the case $k = 1$, we have a solution*

$$u(x) = \frac{1}{2}(1 + |x|)e^{-|x|}$$

Verify that every function of the form

$$v(x) = \frac{1}{2}(1 + |x|)e^{-|x|} + c_1 e^{-x} + c_2 e^{x}$$

is a solution. For which values of c_1 and c_2 does the function decay towards zero as x goes to $\pm\infty$?

- *Verify that, in the case $k \neq 1$, we have a solution*

$$u(x) = -\frac{e^{-k|x|}}{k(k^2 - 1)} + \frac{e^{-|x|}}{k^2 - 1}$$

Verify that every function of the form

$$v(x) = -\frac{e^{-k|x|}}{k(k^2 - 1)} + \frac{e^{-|x|}}{k^2 - 1} + c_1 e^{-kx} + c_2 e^{kx}$$

is a solution.

Solution 5

- *Consider the function*

$$v(x) = \frac{1}{2}(1 + |x|)e^{-|x|} + c_1 e^{-x} + c_2 e^{x}.$$

Obviously, $u(x)$ is a special case of a function of that form when $c_1 = c_2 = 0$. This function is continuous and it is differentiable over $(-\infty, 0)$ and $(0, \infty)$. Its derivative equals

$$v'(x) = -\frac{1}{2}xe^{-|x|} - c_1 e^{-x} + c_2 e^{x}.$$

To see that, we can, e.g., compute the v' for $x > 0$ and $x < 0$ and verify that v' matches this description.

This function is again continuous and it is differentiable over $(-\infty, 0)$ and $(0, \infty)$. Its derivative equals

$$v''(x) = \frac{1}{2}e^{-|x|}(|x| - 1) + c_1 e^{-x} + c_2 e^{x}.$$

That this is a solution to the differential equations is evident from

$$\begin{aligned} -v''(x) + v(x) &= -\frac{1}{2}e^{-|x|}(|x| - 1) - c_1 e^{-x} - c_2 e^{x} + \frac{1}{2}(1 + |x|)e^{-|x|} + c_1 e^{-x} + c_2 e^{x} \\ &= \frac{1}{2}e^{-|x|}(1 - |x|) + \frac{1}{2}(1 + |x|)e^{-|x|} = e^{-|x|}. \end{aligned}$$

To ensure that the solution decays towards zero as x goes to $\pm\infty$, we need to have $c_1 = c_2 = 0$.

- *We repeat the same type of arguments. Consider the function*

$$v(x) = -\frac{e^{-k|x|}}{k(k^2-1)} + \frac{e^{-|x|}}{k^2-1} + c_1 e^{-kx} + c_2 e^{kx}.$$

Obviously, $u(x)$ is a special case of a function of that form when $c_1 = c_2 = 0$. Clearly, v is continuous and it is differentiable over $(-\infty, 0)$ and $(0, \infty)$. Calculations, for $v \neq 0$ show that the derivative (in the sense of distributions) equals

$$v'(x) = \operatorname{sign}(x) \frac{\left(e^{-k|x|} - e^{-|x|}\right)}{k^2-1} + (-k)c_1 e^{-kx} + k c_2 e^{kx}.$$

This function is still continuous and obviously differentiable over $(-\infty, 0)$ and $(0, \infty)$. We find that

$$\begin{aligned} v''(x) &= 2\delta_0 \cdot \frac{\left(e^{-k|x|} - e^{-|x|}\right)}{k^2-1} + \operatorname{sign}(x)^2 \frac{\left(-ke^{-k|x|} + e^{-|x|}\right)}{k^2-1} + k^2 c_1 e^{-kx} + k^2 c_2 e^{kx} \\ &= \frac{\left(-ke^{-k|x|} + e^{-|x|}\right)}{k^2-1} + k^2 c_1 e^{-kx} + k^2 c_2 e^{kx}. \end{aligned}$$

Here, we have used that the $e^{-|x|} - e^{-k|x|} = 0$ at $x = 0$.

That being settled, we check

$$\begin{aligned} &-v''(x) + k^2 v(x) \\ &= -\frac{-ke^{-k|x|} + e^{-|x|}}{k^2-1} - k^2 c_1 e^{-x} - k^2 c_2 e^{x} - k^2 \frac{e^{-k|x|}}{k(k^2-1)} + k^2 \frac{e^{-|x|}}{k^2-1} + c_1 e^{-x} + c_2 e^{x} \\ &= \frac{ke^{-k|x|} - e^{-|x|}}{k^2-1} - \frac{ke^{-k|x|}}{(k^2-1)} + k^2 \frac{e^{-|x|}}{k^2-1} \\ &= e^{-|x|}. \end{aligned}$$

This is the desired differential equation.

Exercise 6 *We want to find a solution to the boundary value problem*

$$\begin{gathered} -\Delta u(x) + k^2 u(x) = x, \quad 0 < x < L, \\ u(0) = 0, \quad u(L) = 0. \end{gathered}$$

- *Extend the right-hand side $f(x) = x$ to an odd function with period $2L$ and compute its Fourier coefficients.*

- *Using these coefficients, find the Fourier series of the solution u. Verify that the boundary condition $u(0) = u(L) = 0$ is satisfied.*

Solution 6 • *The odd extension of period 2 is simply given by*

$$f_{\text{odd}}(x) = x, \quad x \in [-L, L].$$

- *Since f_{odd} is odd by construction, we have*

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} f_n \sin(\frac{n\pi x}{L}),$$

with

$$\begin{aligned}
f_n = \frac{2}{L}\int_0^L x\sin(\frac{n\pi x}{L})\ dx &= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{x=0}^{x=L} + \frac{2}{L}\int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\ dx \\
&= \left[-x\frac{2}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{x=0}^{x=L} + \frac{2}{n\pi}\int_0^L \cos\left(\frac{n\pi}{L}x\right)\ dx \\
&= \left[-x\frac{2}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{x=0}^{x=L} + \frac{2}{n\pi}\left[\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi}{L}x\right)\right]_{x=0}^{x=L} \\
&= \left[-x\frac{2}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{x=0}^{x=L} \\
&= -L\frac{2}{n\pi}\cos(n\pi) = (-1)^{n+1}\frac{2L}{\pi n}.
\end{aligned}$$

- *For the solution, we make a sine ansatz $u(x) = \sum_{n=1}^{\infty} u_n \sin(\frac{n\pi x}{L})$ because this satisfies the boundary conditions. Then, the PDE can be written in terms of the Fourier coefficients as*

$$\left[k^2 + \left(\frac{\pi n}{L}\right)^2\right] u_n = (-1)^{n+1}\frac{2L}{\pi n}, \qquad \forall n \in \mathbb{N}.$$

Therefore, the solution is given by

$$u_n = (-1)^{n+1}\frac{2L}{\pi n}\left[k^2 + \left(\frac{\pi n}{L}\right)^2\right]^{-1}.$$

Exercise 7 (Fun with Neumann boundary conditions) *Consider the Poisson problem with Neumann boundary conditions over the interval $[a, b] = [0, 1]$:*

$$\begin{gathered}
-u''(x) + k^2 u(x) = x - \frac{1}{2}, \quad a < x < b, \\
u'(a) = 0, \quad u'(b) = 0,
\end{gathered}$$

for some $k \geq 0$.

(a) *Extend $f(x) = x - \frac{1}{2}$ to an even function over the real line with period 2.*

(b) *Compute the Fourier coefficients of that even extension of f.*

(c) *Find the Fourier series of a function that satisfies the above differential equation.*

Solution 7

- *The even extension of period 2 is given by* $f_{even}(x) = |x| - \frac{1}{2}, x \in [-1, 1]$.

- *Since f_{even} is even by construction, we only have cosine terms, i.e.*

$$f_{even}(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos(\pi n x).$$

Here,

$$f_0 = \frac{1}{2} \int_{-1}^{1} (|x| - \frac{1}{2}) \, dx = \int_{-1}^{1} (x - \frac{1}{2}) \, dx = 0,$$

and for any $n \geq 1$,

$$\begin{aligned} f_n &= \int_{-1}^{1} (x - \frac{1}{2}) \cos(\pi n x) \, dx \\ &= 2 \int_{0}^{1} (x - \frac{1}{2}) \cos(\pi n x) \, dx = 2 \int_{0}^{1} x \cos(\pi n x) \, dx = \frac{2}{\pi^2 n^2} \left((-1)^n - 1 \right). \end{aligned}$$

- *For the solution, we make a cosine ansatz*

$$u(x) = u_0 + \sum_{n=1}^{\infty} u_n \cos(\pi n x)$$

because the cosine modes satisfy the Neumann boundary conditions. We have expressed the (even extension) of the right-hand side as a Fourier cosine series, and therefore the PDE can be written in terms of the Fourier coefficients:

$$\left[k^2 + \pi^2 n^2 \right] u_n = f_n = \frac{2}{\pi^2 n^2} \left((-1)^n - 1 \right), \qquad \forall n \in \mathbb{N}.$$

Note that no information can be provided for the coefficient u_0 when $k = 0$; indeed, any choice will be possible.[1] *We have $u_0 = 0$.*

This provides the coefficients of the Fourier (cosine) series of u:

$$u_n = \left(k^2 + \pi^2 n^2 \right)^{-1} \frac{2}{\pi^2 n^2} \left((-1)^n - 1 \right), \qquad \forall n \in \mathbb{N}.$$

[1] This corresponds to the fact that whenever we have solution of $-u'' = f$ with Neumann boundary conditions, then also $u + c$ is a solution to the same problem, for any constant $c \in \mathbb{R}$. In other words, the solution is not unique.

Therefore, the solution is given by

$$u(x) = u_0 + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2 \left[k^2 + \pi^2 n^2\right]} \left((-1)^n - 1\right) \cos(\pi n x).$$

The boundary conditions are satisfied by construction, because we have used a Fourier cosine series.

Exercise 8 (Fun with periodic boundary conditions) *Consider the Poisson problem with periodic boundary conditions over the interval $[a, b] = [0, 1]$:*

$$-u''(x) + k^2 u(x) = x - \frac{1}{2}, \quad a < x < b,$$
$$u(a) = u(b),$$

for some $k \geq 0$.

(a) Extend $f(x) = x - \frac{1}{2}$ to a 1-periodic function over the real line.

(b) Compute the Fourier coefficients of that extension of f.

(c) Find the Fourier series of a function that satisfies the above differential equation.

Solution 8 *1. The 1-periodic extension is simply given by $f_{per}(x) = x - \frac{1}{2}, x \in [0, 1]$ with periodic continuation.*

2. When we develop the Fourier coefficients of f_{per},

$$f_{per}(x) = \frac{a_0^f}{2} + \sum_{n\geq 1} a_n^f \cos(2\pi n x) + b_n^f \sin(2\pi n x).$$

We compute the Fourier coefficients. We have

$$a_0^f = 2\int_0^1 (x - \tfrac{1}{2})\ dx = 2\left[\frac{(x - \frac{1}{2})^2}{2}\right]_{x=0}^{x=1} = \left((1 - \tfrac{1}{2})^2 - (0 - \tfrac{1}{2})^2\right) = 0.$$

and

$$\begin{aligned}
a_n^f &= 2\int_0^1 \left(x - \frac{1}{2}\right) \cos(2\pi n x)\ dx \\
&= 2\left[\left(x - \frac{1}{2}\right) \frac{1}{2\pi n} \sin(2\pi n x)\right]_{x=0}^{x=1} - \frac{2}{2\pi n} \int_0^1 \left(x - \frac{1}{2}\right)' \sin(2\pi n x)\ dx \\
&= 2\left[\left(x - \frac{1}{2}\right) \frac{1}{2\pi n} \sin(2\pi n x)\right]_{x=0}^{x=1} - \frac{2}{2\pi n} \int_0^1 \sin(2\pi n x)\ dx
\end{aligned}$$

$$= 0.$$

Notice that $x - \frac{1}{2}$ has zero average, hence $a_0 = 0$. The higher cosine modes of f_{per} do not see the constant, and thus agree with the 1-periodic extension of f, which is odd, and hence the higher cosine modes have coefficient zero as well. We have

$$\begin{aligned} b_n^f &= 2\int_0^1 \left(x - \frac{1}{2}\right) \sin(2\pi n x)\ dx \\ &= 2\int_0^1 x \sin(2\pi n x)\ dx - \int_0^1 \sin(2\pi n x)\ dx \\ &= 2\int_0^1 x \sin(2\pi n x)\ dx. \end{aligned}$$

Via integration by parts:

$$\begin{aligned} b_n^f = 2\int_0^1 x \sin(2\pi n x)\ dx &= 2\left[x\frac{(-\cos)}{2\pi n}(2\pi n x)\right]_{x=0}^{x=1} - \int_0^1 \frac{(-\cos)}{2\pi n}(2\pi n x)\ dx \\ &= -2\frac{1}{2\pi n} + \frac{1}{2\pi n}\int_0^1 \cos(2\pi n x)\ dx = \frac{-1}{\pi n}. \end{aligned}$$

3. *We use a full Fourier series for the solution of the Poisson problem with periodic boundary conditions,*

$$u(x) = \frac{a_0^u}{2} + \sum_{n\geq 1} a_n^u \cos(2\pi n x) + b_n^u \sin(2\pi n x).$$

because both the sine and cosine modes satisfy the periodic boundary conditions. When we write the differential equation in terms of the Fourier coefficients, then we find that a_0 is indeterminate,[2] and

$$\begin{aligned} \left[k^2 + (2\pi n)^2\right] a_n^u &= a_n^f, & \forall n \in \mathbb{N}, \\ \left[k^2 + (2\pi n)^2\right] b_n^u &= b_n^f, & \forall n \in \mathbb{N}. \end{aligned}$$

With our specific choice of coefficients, Therefore, the solution is given by

$$u(x) = \sum_{n=1}^{\infty} \frac{-1}{\pi n\left[k^2 + (2\pi n)^2\right]} \sin(2\pi n x).$$

Note that the periodic boundary conditions are satisfied.

[2]Similar to the Neumann problem, if we have any solution to $-u'' = f$ with periodic boundary conditions, then so does $u + c$ for any constant $c \in \mathbb{R}$.