Analysis III - 203(d)

Winter Semester 2024

Session 13: December 12, 2024

Exercise 1 *The Fourier transform of*

$$f(x) = e^{-5x^2}$$

is the function

$$\hat{f}(\alpha) = \frac{1}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}.$$

Find the Fourier transforms of

$$f', \quad f'', \quad f''', \quad f'''', \quad g(x) = f(2x), \quad h(x) = f(x-3).$$

Solution 1 *To evaluate the Fourier transform of the derivative of a function we will use the following identity from the lecture:*

$$\mathfrak{F}\left(f^{(n)}\right)(\alpha) = (i\alpha)^n \mathfrak{F}(f)(\alpha)$$

We then find

- $\mathfrak{F}(f')(\alpha) = i\alpha\mathfrak{F}(f)(\alpha) = \frac{i\alpha}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$
- $\mathfrak{F}(f'')(\alpha) = -\alpha^2\mathfrak{F}(f)(\alpha) = -\frac{\alpha^2}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$
- $\mathfrak{F}(f''')(\alpha) = -i\alpha^3\mathfrak{F}(f)(\alpha) = \frac{-i\alpha^3}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$
- $\mathfrak{F}(f'''')(\alpha) = \alpha^4\mathfrak{F}(f)(\alpha) = \frac{\alpha^4}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}$

For the Fourier transform of $g(x)$ we use the following identity from the lecture: $g(x) = e^{-ibx} f(ax)$ has the Fourier transform $\hat{g}(\alpha) = \frac{1}{|a|}\hat{f}\left(\frac{\alpha+b}{a}\right)$, consequently:

$$\begin{aligned}
\mathfrak{F}(g(x))(\alpha) &= \mathfrak{F}(f(2x))(\alpha) \\
&= \frac{1}{2}\mathfrak{F}(f(x))\left(\frac{\alpha}{2}\right) \\
&= \frac{1}{2\sqrt{10}} e^{-\frac{(\frac{\alpha}{2})^2}{20}}
\end{aligned}$$

$$= \frac{1}{2\sqrt{10}} e^{-\frac{\alpha^2}{80}}$$

For the Fourier transform of $h(x)$ we use a change of variables:

$$\begin{aligned}
\mathfrak{F}(h(x))(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5(x-3)^2} e^{-\imath x\alpha}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5x^2} e^{-\imath(x+3)\alpha}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5x^2} e^{-\imath x\alpha} e^{-\imath 3\alpha}\, dx \\
&= \frac{e^{-\imath 3\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-5x^2} e^{-\imath x\alpha}\, dx \\
&= e^{-\imath 3\alpha} \mathfrak{F}(f(x))(\alpha) \\
&= \frac{e^{-\imath 3\alpha}}{\sqrt{10}} e^{-\frac{\alpha^2}{20}}. \\
&= \frac{1}{\sqrt{10}} e^{-\frac{\alpha^2}{20} - \imath 3\alpha}.
\end{aligned}$$

Exercise 2 *Find $f : \mathbb{R} \to \mathbb{R}$ such that*

$$\hat{f}(\alpha) = \frac{3}{1+\alpha^2} + \frac{-1}{1+4\alpha^2} + \frac{\sin(4\alpha+3)}{4\alpha+3}$$

Solution 2 *Because of the linearity property we can treat the inverse of each term in seperately. Therefore First,*

$$\begin{aligned}
\mathfrak{F}^{-1}\left(\frac{\sin(4\alpha+3)}{4ar+3}\right)(x) &= \frac{\frac{1}{4}}{\frac{1}{4}} \mathfrak{F}^{-1}\left(\frac{\sin\left(\frac{\alpha+3/4}{\frac{1}{4}}\right)}{\frac{\alpha+\frac{3}{4}}{\frac{1}{4}}}\right)(x) \\
&= \frac{1}{4} e^{-i\frac{3}{4}x} \sqrt{\frac{\pi}{2}} \mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}} \frac{\sin(\alpha)}{\alpha}\right)\left(\frac{x}{4}\right) \\
&= \begin{cases} \frac{1}{4}\sqrt{\frac{\pi}{2}} e^{-i\frac{3}{4}x} & \textit{if } -4 < x < 4 \\ 0 & \textit{else} \end{cases}
\end{aligned}$$

Second,

$$\begin{aligned}
\mathfrak{F}^{-1}\left(\frac{3}{1+\alpha^2}\right)(x) &= 3\sqrt{\frac{\pi}{2}} \mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}} \frac{1}{1+a^2}\right)(x) \\
&= 3\sqrt{\frac{\pi}{2}} e^{-|x|}
\end{aligned}$$

Lastly,

$$\begin{aligned}
\mathfrak{F}^{-1}\left(\frac{-1}{1+4\alpha^2}\right)(x) &= -\sqrt{\frac{\pi}{2}}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+(2\alpha)^2}\right)(x) \\
&= -\sqrt{\frac{\pi}{2}}\frac{\frac{1}{2}}{\frac{1}{2}}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+\left(\frac{\alpha}{\frac{1}{2}}\right)^2}\right)(x \mid \\
&= -\sqrt{\frac{\pi}{2}}\frac{1}{2}\mathfrak{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+\alpha^2}\right)\left(\frac{x}{2}\right) \\
&= -\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-\left|\frac{x}{2}\right|}.
\end{aligned}$$

Altogether this gives

$$f(x) = \begin{cases} -\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-\left|\frac{x}{2}\right|} + 3\sqrt{\frac{\pi}{2}}e^{-|x|} + \frac{1}{4}\sqrt{\frac{\pi}{2}}e^{-i\frac{3}{4}x} & -4 \le x \le 4 \\ -\frac{1}{2}\sqrt{\frac{\pi}{2}}e^{-\left|\frac{x}{2}\right|} + 3\sqrt{\frac{\pi}{2}}e^{-|x|} & \textit{else} \end{cases}$$

Exercise 3 *We consider the Poisson problem with Dirichlet boundary conditions over the interval* $[0, L]$*:*

$$\begin{gathered} -\Delta u(x) = x^2, \quad 0 < x < L, \\ u(0) = 1, \quad u(L) = 2 \end{gathered}$$

- *Solve this problem directly. The solution is a polynomial of order* 4*.*
- *Extend the right-hand side* $f(x) = x^2$ *to an odd function with period* $2L$ *and compute its Fourier coefficients.*
- *Using these coefficients, find the Fourier series of the solution* u^f*, which solves*

$$\begin{gathered} -\Delta u^f(x) = x^2, \quad 0 < x < L, \\ u^f(0) = 0, \quad u^f(L) = 0. \end{gathered}$$

  How do you use the superposition principle to solve the full problem?

Solution 3

- *According to the hint,* $u(x)$ *must be of the form* $u(x) = ax^4 + bx^3 + cx^2 + dx + e$ *for some coefficients* $a, b, c, d, e \in \mathbb{R}$*. Inserting this ansatz into the Poisson equation and the boundary conditions, we find the system of equations*

$$u(0) = e = 1, \tag{1}$$
$$u(L) = aL^4 + bL^3 + cL^2 + dL + e = 2, \tag{2}$$

$$-\Delta u(x) = -12ax^2 - 6bx - 2c = x^2. \tag{3}$$

By matching the coefficients of the polynomials on the right- and left-hand side in (3)*, we find*

$$\begin{aligned} -12a &= 1, \\ -6b &= 0, \\ -2c &= 0. \end{aligned}$$

Hence, we have $a = -\frac{1}{12}, b = 0, c = 0$*. Therefore,* (2) *simplifies to*

$$-\frac{1}{12}L^4 + dL + 1 = 2,$$

from which we deduce that $d = \frac{1}{12}L^3 + \frac{1}{L}$*. Thus, the solution to the Poisson problem is*

$$u(x) = -\frac{1}{12}x^4 + \left(\frac{1}{12}L^3 + \frac{1}{L}\right)x + 1.$$

- *We define the odd extension of* $f(x) = x^2$ *as the function* $\tilde{f}$ *with period* $2L$ *that satisfies*

$$\tilde{f}(x) = \begin{cases} x^2 & \text{if } 0 \le x \le L, \\ -x^2 & \text{if } -L \le x < 0. \end{cases}$$

By construction, $\tilde{f}$ *is an odd function with period* $2L$*. Denote by* $\tilde{a}_n$ *and* $\tilde{b}_n$ *the Fourier coefficients of* $\tilde{f}$*. We immediately have* $\tilde{a}_n = 0$ *for all* $n \in \mathbb{N}$ *because the function is odd. Moreover, by some simple integration by parts*

$$\begin{aligned} \tilde{b}_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\left(\frac{\pi n}{L}x\right)dx = \frac{2}{L}\int_0^L x^2 \sin\left(\frac{\pi n}{L}x\right)dx \\ &= -\frac{2L^2(2 + (-1)^n((n\pi)^2 - 2))}{(n\pi)^3}. \end{aligned}$$

We conclude that

$$\tilde{f}(x) = \sum_{n\ge 1} \tilde{b}_n \sin\left(\frac{\pi n}{L}x\right) = \sum_{n \ge 1} -\frac{2L^2(2 + (-1)^n((n\pi)^2 - 2))}{(n\pi)^3}\sin\left(\frac{\pi n}{L}x\right).$$

- *To apply the superposition principle, we split the problem into two Poisson problems: on the one hand,*

$$\begin{aligned} -\Delta u^f(x) = x^2, &\quad 0 < x < L, \\ u^f(0) = 0, &\quad u^f(L) = 0, \end{aligned}$$

and on the other hand,

$$-\Delta u^g(x) = 0, \quad 0 < x < L,$$
$$u^g(0) = 1, \quad u^g(L) = 2.$$

The full solution will be the sum of u^f *and* u^g*.*

We define

$$u^f(x) = -\sum_{n=1}^{\infty} \frac{\tilde{b}_n}{-\pi^2 n^2/L^2} \sin\left(\pi n x/L\right) \tag{4}$$
$$= \sum_{n=1}^{\infty} \frac{\tilde{b}_n}{\pi^2 n^2/L^2} \sin\left(\pi n x/L\right). \tag{5}$$

Differentation, distributed over the Fourier modes, now shows that

$$-\partial_{xx} u^f(x) = f(x). \tag{6}$$

Additionally, u^f *satisfies the homogeneous Dirichlet boundary conditions.*

We can find u^g *in a manner similar to the previous step. Since its second-derivative is zero,* u^g *must be a linear function over* $[0, L]$*, having the form:*

$$u^g(x) = dx + e. \tag{7}$$

The boundary conditions now lead to $u^g(x) = \frac{1}{L}x + 1$

By the superposition principle, the solution to the original problem is

$$u(x) = u^f(x) + u^g(x). \tag{8}$$

Exercise 4 *Directly compute the solution of the problem*

$$-\Delta u(x) = x^2, \quad 0 < x < L,$$
$$u(0) = 0, \quad u(L) = 0,$$

using elementary analysis. Then find the Fourier coefficients of its odd extension to the interval $[-L, L]$*. Compare this with the function* u^f *from the previous exercise.*

Solution 4 *As in the previous exercise, the solution must be of the form* $u(x) = ax^4 + bx^3 + cx^2 + dx + e$ *for some coefficients* $a, b, c, d, e \in \mathbb{R}$*. Inserting this ansatz into the Poisson equation and the boundary conditions, we find the system of equations*

$$u(0) = e = 0, \tag{9}$$
$$u(L) = aL^4 + bL^3 + cL^2 + dL + e = 0, \tag{10}$$

$$-\Delta u(x) = -12ax^2 - 6bx - 2c = x^2. \tag{11}$$

By matching the coefficients of the polynomials on the right- and left-hand side in (11)*, we find*

$$-12a = 1, \quad -6b = 0, \quad -2c = 0.$$

Hence, we have $a = -\frac{1}{12}$*,* $b = 0$*, and* $c = 0$*. Therefore,* (10) *simplifies to*

$$-\frac{1}{12}L^4 + dL = 0,$$

from which we deduce that $d = \frac{1}{12}L^3$*. Thus, the solution to the Poisson problem is*

$$u(x) = -\frac{1}{12}x^4 + \frac{1}{12}L^3 x.$$

To check that this solution is equal to the one found in the previous exercise, we compute the Fourier coefficients of the odd extension $\tilde{u}$ *of* u *to the interval* $[-L, L]$ *defined by*

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } 0 \leq x \leq L, \\ -u(-x) & \text{if } -L \leq x < 0. \end{cases}$$

By definition, $\tilde{u}$ *is an odd function, and therefore the Fourier series takes the form*

$$\tilde{u}(x) = \sum_{n\geq 1} \tilde{u}_n \sin\left(\frac{\pi n}{L}x\right),$$

where $\tilde{u}_n$ *is the Fourier coefficient of the* n*-th cosine mode. We compute*

$$\begin{aligned}
\tilde{u}_n &= \frac{2}{L}\int_0^L u(x)\sin\left(\frac{\pi n}{L}x\right)dx \\
&= \frac{2}{L}\int_0^L \left(-\frac{1}{12}x^4 + \frac{1}{12}L^3 x\right)\sin\left(\frac{\pi n}{L}x\right)dx \\
&= -\frac{2L^4(2 + (-1)^n)((n\pi)^2 - 2)}{(n\pi)^5}.
\end{aligned}$$

Note that we have $\tilde{u}_n = \left(\frac{L}{\pi n}\right)^2 \tilde{b}_n$*, where* $\tilde{b}_n$ *is as in the previous exercise.*

Exercise 5 *We solve the Poisson problem with homogeneous Dirichlet boundary conditions over* $[0, 1]$*:*

$$-\Delta u(x) = \begin{cases} x & \text{if } 0 < x \leq 0.5 \\ 0 & \text{if } 0.5 < x \leq 1 \end{cases}, \qquad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = 0$$

- *Extend the right-hand side $f(x)$ to an odd function with period 2 and compute its Fourier coefficients.*
- *Using these coefficients, find the solution u. Verify that the boundary condition $u(0) = u(1) = 0$ is satisfied.*

Solution 5

- *We define the odd extension of $f(x)$ as*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \leq x < 1, \\ -f(-x) & \text{if } -1 < x < 0. \end{cases}$$

As in the previous exercises, we find that

$$\tilde{f}(x) = \sum_{n \geq 1} \tilde{f}_n \sin(\pi n x),$$

using that $\tilde{f}$ is odd. The Fourier coefficients of the sine modes are

$$\tilde{f}_n = 2\int_0^1 f(x)\sin(\pi n x)dx = 2\int_0^{\frac{1}{2}} x\sin(\pi n x)dx = \frac{-\pi n\cos(\frac{\pi n}{2}) + 2\sin(\frac{\pi n}{2})}{(\pi n)^2}.$$

One can check that

$$\tilde{f}_n = \begin{cases} \frac{2}{(\pi n)^2}(-1)^n, & \text{if } n \text{ is odd,} \\ -\frac{1}{\pi n}(-1)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

- *We define the solution in terms of its Fourier series:*

$$u(x) = -\sum_{n=1}^{\infty} \frac{\tilde{f}_n}{-\pi^2 n^2}\sin(\pi n x) = \sum_{n=1}^{\infty} \frac{\tilde{f}_n}{\pi^2 n^2}\sin(\pi n x). \tag{12}$$

Clearly, u satisfies the homogeneous Dirichlet boundary conditions, and differentiation provides the desired differential equation

$$-u''(x) = f(x). \tag{13}$$

This shows that u solves the Poisson problem. Note that u is not in the form of a Fourier series over the interval $[0, L]$ but in the form of the Fourier series of its odd extension over $[-L, L]$. The boundary conditions are satisfied because u is a Fourier sine series.

Exercise 6 *Consider the following Fourier sine series for two functions $u, f : \mathbb{R} \to \mathbb{R}$ with period T:*

$$u(x) = \sum_{n=1}^{\infty} b_n^u \sin\left(\frac{2\pi n}{T}x\right), \quad f(x) = \sum_{n=1}^{\infty} b_n^f \sin\left(\frac{2\pi n}{T}x\right).$$

Express the Fourier coefficients of f by the Fourier coefficients of u, and vice-versa, express the Fourier coefficients of u by the Fourier coefficients of f, if u and f are related by the following differential equations:

(a) $-u''(x) = f(x)$.

(b) $u''''(x) = f(x)$.

(c) $-u''(x) + 25u(x) = f(x)$.

(d) $u''''(x) - u''(x) = f(x)$.

(e) $-u''(x) + \gamma \cdot u''(x) = f(x)$.

In the last item, $\gamma \in \mathbb{R}$ is a constant. For which values of γ can you always find a solution?

Solution 6

- *We have*

$$-u''(x) = -\sum_{n=1}^{\infty} b_n^u \left(\frac{2\pi n}{T}\right)^2 \sin\left(\frac{2\pi n}{T} x\right).$$

  Matching coefficients leads to the relationships

$$b_n^u \left(\frac{2\pi n}{T}\right)^2 = b_n^f, \quad b_n^u = \left(\frac{2\pi n}{T}\right)^{-2} b_n^f.$$

- *We have*

$$u''''(x) = \sum_{n=1}^{\infty} b_n^u \left(\frac{2\pi n}{T}\right)^4 \sin\left(\frac{2\pi n}{T} x\right).$$

  Matching coefficients leads to the relationships

$$b_n^u \left(\frac{2\pi n}{T}\right)^4 = b_n^f, \quad b_n^u = \left(\frac{2\pi n}{T}\right)^{-4} b_n^f.$$

- $$-u''(x) + 25u = \sum_{n=1}^{\infty} b_n^u \left(-\left(\frac{2\pi n}{T}\right)^2 + 25\right) \sin\left(\frac{2\pi n}{T} x\right).$$

  Matching coefficients leads to the relationships

$$b_n^u \left(-\left(\frac{2\pi n}{T}\right)^2 + 25\right) = b_n^f, \quad b_n^u = \left(-\left(\frac{2\pi n}{T}\right)^2 + 25\right)^{-1} b_n^f.$$

- $$u''''(x) - u''(x) = \sum_{n=1}^{\infty} b_n^u \left(\left(\frac{2\pi n}{T}\right)^4 - \left(\frac{2\pi n}{T}\right)^2\right) \sin\left(\frac{2\pi n}{T} x\right).$$

  Matching coefficients leads to the relationships

$$b_n^u \left(\left(\frac{2\pi n}{T}\right)^4 - \left(\frac{2\pi n}{T}\right)^2\right) = b_n^f, \quad b_n^u = \left(\left(\frac{2\pi n}{T}\right)^4 - \left(\frac{2\pi n}{T}\right)^2\right)^{-1} b_n^f.$$

- $$u''''(x) - \gamma \cdot u''(x) = \sum_{n=1}^{\infty} b_n^u \left(\left(\frac{2\pi n}{T}\right)^4 - \gamma \cdot \left(\frac{2\pi n}{T}\right)^2 \right) \sin\left(\frac{2\pi n}{T} x\right).$$

  Matching coefficients leads to the relationships

  $$b_n^u \left(\left(\frac{2\pi n}{T}\right)^4 - \gamma \cdot \left(\frac{2\pi n}{T}\right)^2 \right) = b_n^f, \quad b_n^u = \left(\left(\frac{2\pi n}{T}\right)^4 - \gamma \cdot \left(\frac{2\pi n}{T}\right)^2 \right)^{-1} b_n^f.$$

Remark: the second-order equation $-u'' = f$, the Poisson problem, appears in many applications. The fourth-order equation $u'''' = f$ appears in the mathematical theory of elasticity, for example, when modeling the bending of a one-dimensional stick under external force.

Exercise 7 *Recall the definition of the Fourier transform:*

$$\mathfrak{F}[f](\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-it\alpha} \, dt \tag{14}$$

Let $a, b, c \in \mathbb{R}$ be real parameters with $a \neq 0$. Compute the Fourier transforms of the following:

$$g(t) = f(at), \tag{15}$$
$$h(t) = e^{-itb} f(t), \tag{16}$$
$$m(t) = f(t - c). \tag{17}$$

Hint: you have the first two in the lecture and in textbook. You can compute them using results from the lecture or via some standard integral manipulations.

Solution 7 *We write $\hat{f}(\alpha) = \mathfrak{F}[f](\alpha)$ for the Fourier transform of α. To compute the Fourier transforms of the given functions $g(t)$, $h(t)$, and $m(t)$, we proceed as follows:*

- *Let $\hat{g}(\alpha) = \mathfrak{F}[g](\alpha)$. By definition,*

  $$\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-it\alpha} \, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(at) e^{-it\alpha} \, dt.$$

  We perform a substitution $u = at$, so $du = |a| \, dt$ and $t = \frac{u}{a}$. This gives

  $$\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) e^{-i\frac{u}{a}\alpha} \frac{du}{a} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a} \int_{-\infty}^{+\infty} f(u) e^{-i\frac{\alpha}{a}u} \, du.$$

  Recognizing the integral as the Fourier transform of $f(t)$, we find:

  $$\hat{g}(\alpha) = \frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right),$$

- *Let $\hat{h}(\alpha) = \mathfrak{F}[h](\alpha)$. Using definitions.*

$$\begin{aligned}\hat{h}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(t)e^{-it\alpha}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ibt} f(t) e^{-it\alpha}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-it(\alpha+b)}\, dt.\end{aligned}$$

We recognizing the integral as the Fourier transform of $f(t)$. Thus,

$$\hat{h}(\alpha) = \hat{f}(\alpha + b).$$

- *Let $\hat{m}(\alpha) = \mathfrak{F}[m](\alpha)$. By definition,*

$$\hat{m}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} m(t)e^{-it\alpha}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t-c)e^{-it\alpha}\, dt.$$

We substitute $u = t - c$, so $du = dt$ and $t = u + c$. This gives

$$\hat{m}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-i(u+c)\alpha}\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-iu\alpha}e^{-ic\alpha}\, du.$$

We pull out the factor out $e^{-ic\alpha}$,

$$\hat{m}(\alpha) = e^{-ic\alpha} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u)e^{-iu\alpha}\, du,$$

and recognize the last integral as the Fourier transform of $f(t)$. In summary,

$$\hat{m}(\alpha) = e^{-ic\alpha}\hat{f}(\alpha).$$