

Analysis III - 203(d)

Winter Semester 2024

Session 12: December 5, 2024

Exercise 1 Consider the following functions with period T :

- A function f with period $T = 1$ such that

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 0.5 \\ 0 & \text{if } 0.5 \leq x \leq 1 \end{cases}$$

- A function g with period $T = 2\pi$ such that

$$g(x) = \begin{cases} x & \text{if } 0 \leq x < \pi \\ 2\pi - x & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

- A function h with period $T = 1$ such that

$$h(x) = -x \text{ if } 0 \leq x < 1.$$

Find the Fourier coefficients of the Fourier series of these functions. You can use the Fourier series seen in the lecture to get the coefficients.

Solution 1 • As seen in the lecture we know the Fourier coefficients of a square wave,

$$l(x) = \begin{cases} 1, & \text{if } 0 \leq x < 0.5 \\ -1, & \text{if } 0.5 \leq x < 1 \end{cases},$$

are given by:

$$a_n = 0 \text{ for } n \geq 0, \quad b_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

We can express the function $f(x)$ in terms of $l(x)$ as follows:

$$f(x) = \frac{1}{2} + \frac{1}{2}l(x)$$

therefore the Fourier coefficients of f are given by:

$$a_n = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases}, \quad b_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

- As seen in the lecture we know the Fourier coefficients of a triangle wave,

$$m(x) = \begin{cases} 2x, & \text{if } 0 \leq x < 0.5 \\ 2(1-x), & \text{if } 0.5 \leq x < 1 \end{cases}$$

are given by:

$$a_0 = 1, \quad a_n = \begin{cases} -\frac{4}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}, \quad b_n = 0 \text{ for } n > 0$$

We can express the function $g(x)$ in terms of $m(x)$ as follows:

$$g(x) = \pi m\left(\frac{x}{2\pi}\right)$$

therefore the Fourier coefficients of g are given by:

$$a_0 = \pi, \quad a_n = \begin{cases} -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}, \quad b_n = 0 \text{ for } n > 0$$

- As seen in the lecture we know the Fourier coefficients of a sawtooth wave,

$$n(x) = x \text{ for } 0 \leq x < 1$$

are given by:

$$a_0 = 1, \quad a_n = 0 \text{ for } n > 0, \quad a_n = -\frac{1}{n\pi} \text{ for } n < 0,$$

We can express the function $h(x)$ in terms of $n(x)$ as follows:

$$h(x) = -n(x)$$

therefore the Fourier coefficients of h are given by:

$$a_0 = -1, \quad a_n = 0 \text{ for } n > 0, \quad a_n = \frac{1}{n\pi} \text{ for } n < 0,$$

Exercise 2 Explicitly write down the coefficients a_n and b_n and the periods of the following Fourier series:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \sin(2\pi(2n+1)x), \\ g(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} \cos(2\pi(2n-1)x), \\ h(x) &= \frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(\pi nx). \end{aligned}$$

Determine the Fourier coefficients of the derivatives of those functions.

Solution 2 • We substitute $m = 2n + 1$ to obtain

$$f(x) = \sum_{\substack{m=3 \\ m \text{ is odd}}}^{\infty} \frac{1}{m^2} \sin(2\pi mx),$$

therefore the Fourier coefficients are given by

$$a_m = 0 \text{ for } m \geq 0, \quad b_m = \begin{cases} \frac{1}{m^2}, & \text{if } m \text{ is odd and } m \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

and the period is $T = 1$. The derivative of $f(x)$ is given by:

$$f'(x) = \sum_{\substack{m=3 \\ m \text{ is odd}}}^{\infty} \frac{2\pi}{m} \sin(2\pi mx),$$

therefore the Fourier coefficients of the derivative of $f(x)$ are given by

$$b_m = 0 \text{ for } m \geq 0, \quad a_m = \begin{cases} \frac{2\pi}{m}, & \text{if } m \text{ is odd and } m \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

• We substitute $m = 2n - 1$ to obtain

$$g(x) = \sum_{\substack{m=1 \\ m \text{ is odd}}}^{\infty} \frac{(-1)^{\frac{m+1}{2}}}{m^3} \cos(2\pi mx),$$

therefore the Fourier coefficients are given by

$$b_m = 0 \text{ for } m > 0, \quad a_m = \begin{cases} \frac{(-1)^{\frac{m+1}{2}}}{m^3}, & \text{if } m \text{ is odd and } m \geq 1 \\ 0, & \text{if } m \text{ is even and } m \geq 1 \\ 0 & \text{if } m = 0 \end{cases}$$

and the period is $T = 1$. The derivative of $g(x)$ is given by:

$$g'(x) = \sum_{\substack{m=1 \\ m \text{ is odd}}}^{\infty} -\frac{2\pi(-1)^{\frac{m+1}{2}}}{m^2} \sin(2\pi mx),$$

therefore the Fourier coefficients of the derivative of $g(x)$ are given by

$$a_m = 0 \text{ for } m \geq 0, \quad b_m = \begin{cases} -\frac{2\pi(-1)^{\frac{m+1}{2}}}{m^2}, & \text{if } m \text{ is odd and } m \geq 1 \\ 0, & \text{if } m \text{ is even and } m \geq 1 \\ 0 & \text{if } m = 0 \end{cases}$$

- the Fourier coefficients are given by

$$b_n = 0 \text{ for } n > 0, \quad a_n = \begin{cases} \frac{2\pi}{3}, & \text{if } n = 0 \\ -\frac{(-1)^n}{n^2}, & \text{if } n > 1 \end{cases}$$

and the period $T = 2$. The derivative of $h(x)$ is given by:

$$h'(x) = \sum_{n=1}^{\infty} -\frac{\pi(-1)^{n+1}}{n} \sin(\pi nx),$$

therefore the Fourier coefficients of the derivative of $h(x)$ are given by

$$a_n = 0 \text{ for } n \geq 0, \quad b_n = -\frac{\pi(-1)^{n+1}}{n}, \quad \text{for } n > 0$$

Exercise 3 Let $f(x)$ be a periodic function with period T , represented by its Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right) \right).$$

As explained in the lecture,

$$\int_{x_0}^x f(x) dx = \frac{a_0}{2}(x - x_0) + \sum_{n=1}^{\infty} a_n \int_{x_0}^x \cos\left(\frac{2\pi nx}{T}\right) dx + b_n \int_{x_0}^x \sin\left(\frac{2\pi nx}{T}\right) dx.$$

Complete this discussion and find the Fourier series of a function $g(x)$ such that

$$\int_{x_0}^x f(x) = cx + g(x)$$

for some $c \in \mathbb{R}$.

Solution 3 We first compute

$$\int_{x_0}^x \frac{a_0}{2} dt = \frac{a_0}{2}(x - x_0).$$

We integrate the sine and cosine terms, using the fundamental theorem of calculus:

$$\int_{x_0}^x \cos\left(\frac{2\pi nt}{T}\right) dt = \frac{T}{2\pi n} \left(\sin\left(\frac{2\pi nx}{T}\right) - \sin\left(\frac{2\pi nx_0}{T}\right) \right).$$

$$\int_{x_0}^x \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{T}{2\pi n} \left(\cos\left(\frac{2\pi nx}{T}\right) - \cos\left(\frac{2\pi nx_0}{T}\right) \right).$$

Combining all terms, the integral $F(x)$ becomes:

$$F(x) = \frac{a_0}{2}(x - x_0) + \sum_{n=1}^{\infty} a_n \frac{T}{2\pi n} \left(\sin\left(\frac{2\pi nx}{T}\right) - \sin\left(\frac{2\pi nx_0}{T}\right) \right) - b_n \frac{T}{2\pi n} \left(\cos\left(\frac{2\pi nx}{T}\right) - \cos\left(\frac{2\pi nx_0}{T}\right) \right).$$

We rewrite this once again and obtain

$$\begin{aligned} F(x) = \frac{a_0}{2}x - \frac{a_0x_0}{2} + \sum_{n=1}^{\infty} \left((-a_n) \frac{T}{2\pi n} \sin\left(\frac{2\pi nx_0}{T}\right) + b_n \frac{T}{2\pi n} \cos\left(\frac{2\pi nx_0}{T}\right) \right) \\ + \sum_{n=1}^{\infty} (-b_n) \frac{T}{2\pi n} \cos\left(\frac{2\pi nx}{T}\right) + a_n \frac{T}{2\pi n} \sin\left(\frac{2\pi nx}{T}\right). \end{aligned}$$

We thus obtain $c = \frac{a_0}{2}$ and the Fourier coefficients of the function g :

$$\begin{aligned} \frac{A_0}{2} &:= -\frac{a_0x_0}{2} + \sum_{n=1}^{\infty} \left((-a_n) \frac{T}{2\pi n} \sin\left(\frac{2\pi nx_0}{T}\right) + b_n \frac{T}{2\pi n} \cos\left(\frac{2\pi nx_0}{T}\right) \right) \\ A_n &:= (-b_n) \frac{T}{2\pi n} \\ B_n &:= a_n \frac{T}{2\pi n} \end{aligned}$$

This completes the discussion.

Exercise 4 Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are functions. Recall that a function is called even if

$$f(-x) = f(x)$$

and odd if

$$f(-x) = -f(x)$$

- Show that if f and g are both odd or both even, then fg is even
- Show that if one of f and g is odd and the other is even, then fg is odd.
- Show that the only function that is both odd and even has constant value zero.

Solution 4 • If f and g are both even, then

$$f(-x)g(-x) = f(x)g(x),$$

and if both are odd, then

$$f(-x)g(-x) = (-1)^2 f(x)g(x) = f(x)g(x),$$

so fg must be an even function.

- Suppose that f is even and that g is odd. Then

$$f(-x)g(-x) = f(x)g(-x) = -f(x)g(x),$$

showing that fg is an odd function. The same is true if we switch the role of f and g .

- Suppose that f is both odd and even. Then

$$f(x) = f(-x) = -f(x)$$

for each $x \in \mathbb{R}$. But the only number that equals its negative is zero. So $f(x) = 0$ for each x .

Exercise 5 Compute the Fourier transform of the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

You can either directly use the complex exponential, or you can express it in terms of the sine and cosine function.

(Interpretation: the function $f(x)$ describes a localized signal: it is zero at $x = 0$, then it rises linearly up to 1, and then it jumps back to zero and remains zero from there on. The signal is not periodic.)

Solution 5 We write down the solution in two different ways, either using the complex exponential directly, or writing it as a sum of sine and cosine.

$$\begin{aligned} \mathfrak{F}(f)(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^1 x e^{-i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{x}{-i\alpha} e^{-i\alpha x} \right]_0^1 - \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{-i\alpha} e^{-i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{x}{-i\alpha} e^{-i\alpha x} \right]_0^1 + \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{i\alpha} e^{-i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{x}{-i\alpha} e^{-i\alpha x} \right]_0^1 + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha^2} e^{-i\alpha x} \right]_0^1 \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{-i\alpha} e^{-i\alpha} + \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\alpha^2} e^{-i\alpha} - \frac{1}{\alpha^2} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{\alpha} e^{-i\alpha} + \frac{1}{\alpha^2} e^{-i\alpha} - \frac{1}{\alpha^2} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{\alpha} \cos \alpha + \frac{1}{\alpha} \sin \alpha + \frac{1}{\alpha^2} \cos \alpha - \frac{i}{\alpha^2} \sin \alpha - \frac{1}{\alpha^2} \right) \end{aligned}$$

Next we do it in terms of sine and cosine functions.

$$\begin{aligned}
\mathfrak{F}(f)(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (\cos \alpha x - i \sin \alpha x) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \alpha x dx - i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \sin \alpha x dx
\end{aligned}$$

We evaluate each integral separately.

$$\begin{aligned}
\int_0^1 x \cos(\alpha x) dx &= \left[\frac{x}{\alpha} \sin \alpha x \right]_0^1 - \int_0^1 \frac{1}{\alpha} \sin \alpha x dx \\
&= \left[\frac{x}{\alpha} \sin \alpha x \right]_0^1 + \left[\frac{1}{\alpha^2} \cos \alpha x \right]_0^1 \\
&= \frac{1}{\alpha} \sin \alpha + \frac{1}{\alpha^2} \cos \alpha - \frac{1}{\alpha^2}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x \sin(\alpha x) dx &= \left[-\frac{x}{\alpha} \cos(\alpha x) \right]_0^1 + \int_0^1 \frac{1}{\alpha} \cos(\alpha x) dx \\
&= \left[-\frac{x}{\alpha} \cos(\alpha x) \right]_0^1 + \left[\frac{1}{\alpha^2} \sin \alpha x \right]_0^1 \\
&= -\frac{1}{\alpha} \cos \alpha + \frac{1}{\alpha^2} \sin \alpha
\end{aligned}$$

All together we have:

$$\begin{aligned}
\mathfrak{F}(f)(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\alpha} \sin \alpha + \frac{1}{\alpha^2} \cos \alpha - \frac{1}{\alpha^2} + i \frac{1}{\alpha} \cos \alpha - i \frac{1}{\alpha^2} \sin \alpha \right)
\end{aligned}$$

Exercise 6 Find the Fourier transform of

$$f(x) = \begin{cases} \sin(x) & \text{if } 0 \leq x \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Solution 6 We try to perform integration by parts

$$\mathcal{F}(f(x))(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \left[\sin x \frac{e^{-i\alpha x}}{-i\alpha} \right]_0^{2\pi} + \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{\cos x}{i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{\cos x}{i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{\cos x}{\alpha^2} e^{-i\alpha x} \right]_0^{2\pi} + \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{\sin x}{\alpha^2} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}\alpha^2} (e^{-2\pi i\alpha} - 1) + \frac{1}{\alpha^2} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx
\end{aligned}$$

It seems we have obtained the term that we started with. However,

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}\alpha^2} (e^{-2\pi i\alpha} - 1) + \frac{1}{\alpha^2} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx$$

can be rearranged to

$$\left(1 - \frac{1}{\alpha^2}\right) \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}\alpha^2} (e^{-2\pi i\alpha} - 1).$$

It follows that

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sin x e^{-i\alpha x} dx &= \left(1 - \frac{1}{\alpha^2}\right)^{-1} \frac{1}{\sqrt{2\pi}\alpha^2} (e^{-2\pi i\alpha} - 1) \\
&= \left(\frac{\alpha^2 - 1}{\alpha^2}\right)^{-1} \frac{(e^{-2\pi i\alpha} - 1)}{\sqrt{2\pi}\alpha^2} \\
&= \frac{\alpha^2}{\alpha^2 - 1} \frac{(e^{-2\pi i\alpha} - 1)}{\alpha^2 \sqrt{2\pi}} \\
&= \frac{(e^{-2\pi i\alpha} - 1)}{\sqrt{2\pi}(\alpha^2 - 1)}
\end{aligned}$$

which is the desired Fourier transform.

Exercise 7 (Extra) We have introduced the Fourier transform

$$\mathfrak{F}(f)(\alpha) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

Different authors define the Fourier transform alternatively by:

$$\mathfrak{F}_2(f)(\xi) := \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx, \quad \mathfrak{F}_3(f)(\omega) := \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Express $\mathfrak{F}_2(f)$ and $\mathfrak{F}_3(f)$ in terms of $\mathfrak{F}(f)$.

Solution 7 Obviously,

$$\mathfrak{F}_3(f)(\omega) = \sqrt{2\pi} \mathfrak{F}(f)(\omega).$$

For the other transformation, we write:

$$\mathfrak{F}_2(f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx = \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx = \sqrt{2\pi} \cdot \mathfrak{F}(f)(2\pi\xi).$$